EXPLORING MODERN MATHEMATICS

BOOK 2

CURRICULUM

KEEDY · JAMESON · JOHNSON

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BOOK 2

EXPLORING MODERN MATHEMATICS

KEEDY • JAMESON • JOHNSON

HOLT, RINEHART AND WINSTON, INC., NEW YORK

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Preface

This is book 2 of a three-book series of modern mathematics textbooks. The series was inspired by and is based upon the work of the University of Maryland Mathematics Project (Jr.H.S.) and the School Mathematics Study Group.

This series has been designed to provide the basis of a junior high school curriculum, to fit the increasingly prevalent 6-3-3 school organization. Such a curriculum, based on this series of textbooks, would include such features as the following:

1. Integration of arithmetic and algebra. These subjects are not separated, and thus more than one year's time is devoted to algebra. In this book, for example, topics which are traditionally known as algebraic are to be found throughout, besides which there is one chapter devoted to solving equations and another to polynomials.
2. Integration of geometry with arithmetic and algebra where feasible. In this book, for example, there are two substantial geometry chapters, and geometric notions are used in other places as well, particularly in connection with graphs.
3. Gradually increasing rigor. The proofs required in this book are somewhat more demanding than those in book 1, and also somewhat more profound. The writing of formal proofs begins in book 3.
4. Increasing emphasis on applied problems. In book 1 relatively few applications are considered, whereas in this book there are considerably more. In book 3 the number and complexity of these kinds of problems increases still more.
5. Increasing attention to reading. There are more sections of this book suitable for reading assignments than in book 1. In book 3 there are still more. The reading and vocabulary level also gradually increases throughout the series.
6. Opportunity for the student to explore and discover for himself.

The exploratory exercises, headed *Let's Explore*, provide appropriate experiences, leading to discovery or insight. These sections allow for *individual* student experience, so necessary for discovery. It is because of this feature of the books that they are called "semi-programed." Supplements are available in which the exploratory sections are more fully programed.

Although these books have been written for a three-year junior high school sequence for average students, they might well find other uses, and that possibility has been kept in mind in the writing. They would provide adequate and interesting material, for example, for a one-, two- or three-year sequence in general mathematics in the senior high school.

Classroom tests of this material have been extremely valuable. Considerable improvements resulted from the excellent suggestions made by teachers testing the material in their own classes. For this help special thanks go to Mr. Joseph Ciechon at Westport, Conn.; Mrs. Margaret Fleming, Mrs. Jane Price and Miss Lynn MacKenzie in Montgomery County, Md. Acknowledgement is made to the many people who made contributions to the UMMaP and SMSG projects, the ideas of which have influenced the authors so greatly. Also, thanks are given to *Scripta Mathematica* for permission to reprint the portraits of famous mathematicians.

For further details about the spirit of the series and suggestions as to how it may be used, the reader is referred to the *Teacher Edition*.

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About This Book

If the last mathematics textbook you used had a title different from this one, you will probably find this book different. In studying from this book, you will be expected to use your own imagination. You will learn to see patterns, make guesses and then prove whether or not your guesses are correct. You must not be afraid to guess wrong. Everybody does, sometimes at least. If you guess wrong, you will then figure out why you were wrong. This is a good way to learn and it can be a lot of fun.

If the last mathematics textbook you used was called *Exploring Modern Mathematics, Book 1*, this book will not seem strange to you. As you study book 2 you will explore more deeply the world of mathematics. As before, you will look for patterns, get hunches and then try to prove or disprove them. Above all, you should always think for yourself.

You will find that more is expected of you, now that you are older and more experienced. You will need to think more deeply about mathematical ideas. You will need to be more careful about proving your hunches about the patterns you see. You will also be expected to learn more about how mathematics is used to work real world problems, and to read mathematics more easily. Day by day you should expect to grow mathematically wiser and more skilled. We sincerely hope you have fun doing it.

M. L. K.
R. E. J.
P. L. J.

CHAPTER ONE

The System of Integers

$$|7 + {}^{-}5| \leq |7| + |{}^{-}5|$$

$$^{-}[^{-}x]$$

$$x = {}^{-}y \rightarrow x + y = 0$$

MATHEMATICAL SYSTEMS

We have already studied the properties of several mathematical systems. The natural numbers are a mathematical system and so are the whole numbers. Other systems we have studied are the number system of arithmetic and the arithmetic of a clock.

In each mathematical system we have a set of numbers or other things. We also have one or more operations. Do you remember what we mean by an *operation?* One example is subtraction in the system of natural numbers. We know that

$$12 - 4 = 8.$$

Given the two numbers 12 and 4 we have a way of finding some natural number for an "answer". There is not more than one answer. Sometimes we cannot subtract. For example,

$$‘5 - 17’$$

is not the name of any natural number. The subtraction is not possible.

Subtraction is an operation however, because whenever there is an "answer" there is not more than one.

Most of the mathematical systems we have studied have somewhat similar properties. Let's review some of the properties that we have seen in several different systems.

1. *Closure.* As an example of this property, the set of natural numbers is closed under addition. This means that for *any* natural numbers m and n, the sum $m + n$ is also in the set of natural numbers. (There is always an "answer" and that answer is a natural number.) The set of natural numbers is not closed under subtraction. Why?

2. *Commutative Property for an Operation.* As an example of this property, multiplication of numbers of arithmetic is commutative. This means that for any numbers of arithmetic $\frac{a}{b}$ and $\frac{c}{d}$,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}.$$

   In other words, the order taken in multiplying does not make any difference.

3. *Associative Property for an Operation.* As an example of this property, addition of whole numbers is associative. This means that for any whole numbers a, b and c,

$$a + (b + c) = (a + b) + c.$$

   In other words, the way we group the addends does not make any difference.

4. *Distributive Property of Multiplication over Addition.* In the system of natural numbers this property holds. For any natural numbers a, b and c,

$$a \cdot (b + c) = a \cdot b + a \cdot c, \text{ and also}$$
$$(b + c) \cdot a = b \cdot a + c \cdot a.$$

5. *Multiplicative Identity.* The natural numbers, whole numbers and numbers of arithmetic all have a multiplicative identity. It is the number 1. This means that if we multiply any number n by 1, we get that number, n.

6. *Multiplicative Inverses.* In the number system of arithmetic all non-zero numbers have multiplicative inverses, or *reciprocals.* Whenever a number is multiplied by its multiplicative inverse,

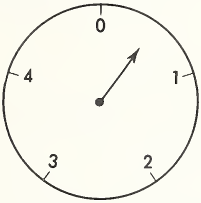
the result is the identity. 1. The reciprocal of 3 is $\frac{1}{3}$, and the reciprocal of $\frac{1}{3}$ is 3.

7. *Additive Identity.* In whole numbers and in the system of arithmetic there is an additive identity. It is the number 0. This means that if we add 0 to any number n, we get that number, n. (Do you remember that we can never divide by 0?)

8. *Additive Inverses.* If we can add two numbers and get the additive identity, 0, for the result, then we say that the numbers are *additive inverses* of each other. In most of the systems we have studied there were not many numbers having additive inverses. We did find them in clock arithmetic.

Since we have not had much experience with additive inverses, let's review them in clock arithmetic. We'll use a five-minute clock. If we add 3 and 2 we get 0.

$$3 + 2 = 0.$$



Therefore 3 and 2 are additive inverses of each other. We can name the additive inverse of 3 like this:

'${}^{-}3$'.

We can name the additive inverse of 2, like this:

'${}^{-}2$'.

To say that the additive inverse of 3 is 2 we may use the sentence

$$\underbrace{{}^{-}3}_{\text{the additive inverse of 3}} \qquad \underbrace{=}_{\text{is}} \qquad \underbrace{2.}_{2}$$

We also know of course that

$$\underbrace{{}^{-}2}_{\text{the additive inverse of 2}} \qquad \underbrace{=}_{\text{is}} \qquad \underbrace{3.}_{3}$$

We also learned about additive inverses of additive inverses. For example,

$${}^{-}({}^{-}3) = 3.$$

(The additive inverse of the additive inverse of 3 is 3.)

We found out, too, about additive inverses of sums. For example

$${}^{-}(1 + 2) = {}^{-}1 + {}^{-}2, \text{ and}$$
$${}^{-}(4 + {}^{-}2) = {}^{-}4 + 2.$$

In general, if x and y are any numbers in clock arithmetic, then

$${}^{-}(x + y) = {}^{-}x + {}^{-}y.$$

You may also remember discovering that

$$3 - 4 \text{ and } 4 - 3$$

are additive inverses of each other. Check this on the clock. Do you remember that if x and y are any numbers in clock arithmetic, then $x - y$ and $y - x$ are additive inverses of each other? That is,

$$x - y = {}^{-}(y - x).$$

It was in clock arithmetic that we used additive inverses to define subtraction. We found that we could subtract by adding an inverse. For example,

$$2 - 4 = 2 + {}^{-}4, \text{ and}$$
$$3 - {}^{-}2 = 3 + 2.$$

In general, if x and y are any numbers in clock arithmetic, then

$$x - y = x + {}^{-}y.$$

We do not have a large number system in which all numbers have additive inverses. It will be useful to make up such a system. Before doing this, you should work some review exercises on mathematical systems.

EXERCISES

1. A mathematical system consists of a set and one or more ____________.
2. When we have a way (or rule) for combining two members of a set, and there is never more than one "answer" we say we have an __________.
3. If for *any* two members of a set the result of an operation is in that set, we say the set is ____________ under the operation.
4. If we add 0 (in the system of whole numbers) to any number n, the result is ____________.
   Therefore we say that 0 is the ____________ ____________.
5. The multiplicative identity in the number system of arithmetic is the number ____________.
6. A multiplicative inverse is also called a ____________.
7. In the number system of arithmetic, which of these are names for the reciprocal of 3?
   a. $3 \cdot 1$ b. $\frac{1}{3}$ c. $3 \div 1$ d. $1 \div 3$
8. In the number system of arithmetic, which of these are names for the reciprocal of $\frac{3}{4}$?
   a. $\frac{4}{3}$ b. $\dfrac{1}{\frac{4}{3}}$ c. $\dfrac{1}{\frac{3}{4}}$ d. $3 \div 4$ e. $4 \div 3$

9. In clock arithmetic the symbol '$^{-}3$' represents the __________ __________ of 3.

10. In clock arithmetic, the symbol '$^{-}(^{-}2)$' is another name for:
a. $^{-}2$ b. 2 c. $\frac{1}{2}$ d. $0 - 2$

11. In clock arithmetic the symbol '$^{-}(3 + 4)$' is another name for:
a. $3 + 4$ b. $^{-}3 + 4$ c. $^{-}3 + {}^{-}4$ d. $3 + {}^{-}4$

12. In clock arithmetic we can subtract by adding an inverse. Which of these equations is an example of this?
a. $3 - 4 = 4 - 3$
b. $4 - 2 = 4 + 2$
c. $4 - 3 = 4 + {}^{-}3$
d. $4 - 3 = {}^{-}4 + 3$

13. In clock arithmetic $2 - 4$ and $4 - 2$ are __________ __________ of each other.

14. Which of these equations illustrates the commutative property of addition?
a. $34 = 43$
b. $5 + 3 = 3 + 5$
c. $5 + {}^{-}3 = 5 - 3$
d. $4.5 = 5.4$

15. The commutative property of addition allows us to change the __________ of the addends in a sum.

16. The __________ property of addition allows us to change the grouping in a sum.

17. Give an example to illustrate the associative property.

Review Practice

1. Add these numbers of arithmetic. (Don't forget to estimate.)

a. $$\begin{array}{r}340\\278\\\underline{534}\end{array}$$

b. $$\begin{array}{r}162\\583\\\underline{272}\end{array}$$

c. $$\begin{array}{r}25342\\16001\\\underline{90003}\end{array}$$

d. $$\begin{array}{r}1.24\\23.5\\.07\\\underline{.009}\end{array}$$

e. $$\begin{array}{r}22.87\\1.0093\\\underline{142.06}\end{array}$$

f. $$\begin{array}{r}1.7362\\245.46\\\underline{92.3}\end{array}$$

2. Subtract. (Don't forget to estimate.)

a. $$\begin{array}{r}2350\\\underline{-1402}\end{array}$$

b. $$\begin{array}{r}2793\\\underline{-142}\end{array}$$

c. $\begin{array}{r} 25203 \\ -20079 \\ \hline \end{array}$

d. $\begin{array}{r} 1.462 \\ -.5936 \\ \hline \end{array}$

e. $\begin{array}{r} 1.53 \\ -.752 \\ \hline \end{array}$

f. $\begin{array}{r} 625.324 \\ -406.213 \\ \hline \end{array}$

g. $\begin{array}{r} 72.19 \\ -46.2789 \\ \hline \end{array}$

h. $\begin{array}{r} 143.897 \\ -97.365 \\ \hline \end{array}$

i. $\begin{array}{r} 219.0072 \\ -30.24 \\ \hline \end{array}$

3. Multiply. (Don't forget to estimate.)

a. $\begin{array}{r} 324 \\ 226 \\ \hline \end{array}$

b. $\begin{array}{r} 142 \\ 253 \\ \hline \end{array}$

c. $\begin{array}{r} 361 \\ 157 \\ \hline \end{array}$

d. $\begin{array}{r} 1.78 \\ 1.02 \\ \hline \end{array}$

e. $\begin{array}{r} 21.98 \\ 22.6 \\ \hline \end{array}$

f. $\begin{array}{r} 10.3 \\ 9.7 \\ \hline \end{array}$

g. $\begin{array}{r} .1057 \\ 673 \\ \hline \end{array}$

h. $\begin{array}{r} 79.846 \\ 1.52 \\ \hline \end{array}$

i. $\begin{array}{r} 32.47 \\ 3.09 \\ \hline \end{array}$

4. Divide. (Don't forget to estimate.)

a. $4\overline{)988}$

b. $12\overline{)288}$

c. $15\overline{)225}$

d. $4\overline{)3.784}$

e. $.04\overline{)9.88}$

f. $2.5\overline{)927.5}$

g. $.012\overline{).0396}$

h. $15\overline{).00225}$

i. $.0009\overline{).369}$

INTEGERS

Now let's make up our new number system. We'll start with the set of whole numbers. The only whole number having an additive inverse is 0. Let's make up additive inverses for all the rest of the whole numbers. We'll name them like this:

‘$^{-}2$’, ‘$^{-}17$’, ‘$^{-}4153$’

We read '$^{-}2$' "the additive inverse of 2." When there is no confusion we can simply say "the inverse of 2," and omit the word *additive*. All the numbers in our new set will be called *integers*.

The set of integers consists of 0, all the natural numbers, and an additive inverse for each natural number.
*The additive inverse of any integer **n** is named '^{-}n'.*

1.1 Let's Explore

1. The additive inverse of 2 is named '$^{-}2$'. What is the simplest name for $2 + {}^{-}2$?
2. The additive inverse of 38 is $^{-}38$. What is the simplest name for $38 + {}^{-}38$?
3. The sentence $14 + {}^{-}14 = 0$ is true. How do we know it is true?
4. The symbol '$^{-}0$' represents what number?
5. What does the symbol '$^{-}12$' represent?
6. What does the symbol '$^{-}(^{-}12)$' represent?
7. Is this sentence true?

$$^{-}(^{-}3) + {}^{-}3 = 0$$

Why?

THE NUMBER LINE FOR INTEGERS

We used points on a ray to represent whole numbers. We also used points on a ray to represent numbers of arithmetic. We can use points on a line to represent integers.

1.2 Let's Explore

1. Draw a ray. Start in the middle of your paper. Draw it to the right.
2. Label the endpoint of the ray '0'.
3. Use a unit this long ______. Label points for whole numbers. Extend the ray to the edge of your paper.
4. Where shall we label points for the additive inverses? Extend your ray to the left, to make a line.

5. One unit to the left of ‘0’ make a dot. Label it ‘$^{-}1$’.

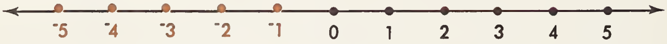
6. One unit to the left of ‘$^{-}1$’ make a dot. Label it ‘$^{-}2$’.

7. Continue this to the edge of your paper.

8. You now have a number line for the integers. If your paper were big enough, how far in each direction would it go?

POSITIVE AND NEGATIVE INTEGERS

Your number line should look like this.



Do you remember how a point divides a line? It separates it into 3 sets. Two of them are half-lines. The other contains just the point itself.

The point 0 divides this line. The numbers named on one half-line are the natural numbers. Those named on the other half-line are their additive inverses. The integers are separated into three sets.

The numbers named on the left are called *negative* integers. Those named on the right are called *positive* integers. Sometimes we name positive integers like this:

‘$^{+}3$’, ‘$^{+}45$’, ‘$^{+}138$’

We read ‘$^{+}3$’ “positive three.” Symbols like these name natural numbers. We name them this way sometimes for emphasis, so that we won’t confuse them with the negative integers.

When we drew number rays for whole numbers and numbers of arithmetic, we placed the numbers *in order*. Points for larger numbers were farther to the right. We could tell that $2 < 6$ because the point for 2 is farther left.

Let’s keep this idea for the integers. We can agree that $^{-}3 < 0$ because the point for $^{-}3$ is to the left of the point for 0. With this idea, we would agree that all these sentences are true.

$5 > 3$, $6 > {}^{-}5$, ${}^{-}5 < 0$, ${}^{-}7 < {}^{-}4$

A sentence $\mathbf{x} < \mathbf{y}$*, for integers, is true if, on a horizontal number line, the point for* $\mathbf{x}$ *is to the left of the point for* $\mathbf{y}$*.*

Do you notice that the negative integers are all less than 0? Of course the positive integers are greater than zero. Check this on the number line.

Integers greater than zero are ***positive.***
Integers less than zero are ***negative.***
Zero is neither positive nor negative.

EXERCISES

1. Name the additive inverse of each integer.
   - a. 2
   - b. 4
   - c. 17
   - d. 35
   - e. 0
   - f. $^{-}3$
   - g. $^{-}8$
   - h. $^{-}16$
   - i. $^{-}27$

2. Name the additive inverse of each of the following integers in two ways.

Example:

The inverse of $^{-}4$ is $^{-}(^{-}4)$ or 4.

- a. $^{-}5$
- b. $^{-}7$
- c. $^{-}27$
- d. $^{-}0$
- e. $^{-}465$
- f. $^{-}376$

3. Draw a number line for integers. Circle each symbol for a positive number. Draw a square around each symbol for a negative number.

4. Draw a number line for integers. Circle each symbol for a number greater than $^{-}3$. Draw a square around each symbol for a number less than $^{-}3$.

5. Which of these sentences for integers are true?
   (Remember that '$\leq$' means "less than or equal to" and '$\geq$' means "greater than or equal to".)
   - a. $^{+}1 < 4$
   - b. $2 \leq {}^{+}6$
   - c. $0 < {}^{+}7$
   - d. $^{-}3 < 0$
   - e. $^{-}3 < {}^{+}1$
   - f. $^{-}5 \leq {}^{+}3$
   - g. $1 < {}^{-}5$
   - h. $^{+}3 < {}^{-}7$
   - i. $^{+}3 > {}^{+}4$
   - j. $^{-}2 \geq {}^{-}5$

6. Which of these sentences are true?
   - a. The additive inverse of a positive integer is negative.
   - b. The additive inverse of a natural number is a negative integer.

c. The additive inverse of a negative integer is sometimes positive and sometimes negative.
d. Natural numbers are also called positive integers.

7. Write the simplest name for each of these integers.
   a. $^{+}5$
   b. $^{-}(^{+}5)$
   c. $^{-}(^{-}17)$
   d. $3 + 4$
   e. $^{+}15 + {}^{+}12$
   f. $8 + {}^{-}8$
   g. $0 + 0$
   h. $^{+}14 + {}^{-}0$
   i. $^{-}358 + {}^{+}358$
   j. $^{-}(^{-}(^{+}3))$
   k. $^{-}(^{-}(^{-}12))$

ABSOLUTE VALUE

How far from the origin is the point for 3? How far from the origin is the point for $^{-}3$? We see that both points are 3 units from the origin.

To tell the distance of a point from the origin we talk of *absolute value.* The absolute value of an integer is this distance. For absolute value we use a symbol '$|m|$'. This is read "the absolute value of m." We understand that 'm' represents an integer. Then '$|m|$' also represents an integer, but it cannot be negative.

1.3 Let's Explore

1. Draw a number line for integers.
   a. Circle all points for which the absolute value is 1.
   b. Draw a square around all points for which the absolute value is 2.
   c. Draw a triangle around all points for which the absolute value is 3.
   d. Draw a rectangle around all points for which the absolute value is 0.
   e. Draw a pentagon around all points for which the absolute value is $^{-}2$.

2. Write the simplest name for each integer.
   a. $|3|$
   b. $|^{-}12|$
   c. $|7|$
   d. $|^{-}35|$
   e. $|^{+}17|$
   f. $|0|$

3. Which sentences are true?
   a. $|5| = |^{-}5|$
   b. $|3| < |^{+}7|$
   c. $|^{+}2| > |^{-}7|$
   d. $|^{-}6| > |^{+}3|$
   e. $|3| + |^{+}2| \leq |3 + 2|$
   f. $|4| - |3| \geq |4 - 3|$

4. Find the additive inverse
   a. $^{-}3$
   b. $^{-}8$
   c. $^{-}5$
   d. $^{-}12$
   e. 4
   f. 5
   g. 17
   h. 25
   i. 0

5. Find the absolute value of each of the following.
   - a. $^-3$
   - b. $^-8$
   - c. $^-5$
   - d. $^-12$
   - e. 4
   - f. 5
   - g. 17
   - h. 25
   - i. 0

Here is a definition of absolute value for integers.

*The **absolute value** of an integer **m** is represented by the symbol '|**m**|'. It is defined as follows:*

$$|m| = m \quad \text{when} \quad m \geq 0, \text{ and}$$
$$|m| = {}^-m \quad \text{when} \quad m < 0.$$

Notice what the last part says. If m is negative then its absolute value is its additive inverse. This means that its absolute value is positive. Remember, the absolute value of any integer is always an integer. Also, *absolute values are never negative.*

EXERCISES

1. Find the simplest name for each of these integers.
   - a. $|{}^-5|$
   - b. $|{}^+4|$
   - c. $|0|$
   - d. $|{}^-0|$
   - e. ${}^-|4|$
   - f. ${}^-|{}^-7|$
2. Which of these sentences are true?
   - a. $|{}^+7| = |{}^-7|$
   - b. $|2| > {}^-135$
   - c. $|{}^+4| + |{}^-5| = {}^+9$
   - d. $|{}^-12| + |{}^+12| = 0$
   - e. $|{}^-8| + {}^-|{}^+8| = 0$
   - f. $|4 - 2| < |4| - |2|$
3. Find the solution set of each sentence in the set of integers.
   - a. $|x| = 3$
   - b. $|y| = {}^+5$
   - c. $|x| = 0$
   - d. $|z| = {}^-3$
   - e. $|x| + 2 = 3$
   - f. $|x| < 5$

ADDITION FOR INTEGERS

We have made up the set of integers. We started with the whole numbers. For each natural number we made up an additive inverse. We already know how to add whole numbers. We already know how to add any integer and its additive inverse. But we have not defined addition in any other case. We shall need to do so. First, let's list the cases in which addition is already defined.

1. $m + n$, where m and n are any whole numbers.
2. $m + {}^{-}m$, where m is any whole number.

Now which cases must we define? Let's list them.

3. $0 + {}^{-}m$, where m is any natural number.
(This is addition of 0 and any negative integer.)
4. ${}^{-}m + {}^{-}n$, where m and n are any natural numbers.
(This is addition of two negative integers.)
5. ${}^{-}m + n$, where m and n are any two different natural numbers.
(This is addition of a positive integer and a negative integer.)

How shall we define addition for cases 3, 4 and 5? Since we are making up the system, we could do it any way we choose. Can we make these definitions so that addition is commutative? Associative? Let's do it if possible. Then most of the rules for adding whole numbers will work for integers too.

ADDING ZERO

Zero is the additive identity for whole numbers. If we want an additive identity in the system of integers how should we define the sum of 0 and a negative integer? We see that we need to agree that $0 + {}^{-}m = {}^{-}m$, for any natural number m. If we want addition for integers to be commutative, then we must also agree that ${}^{-}m + 0 = {}^{-}m$, for any natural number m.

Let's make these definitions, or agreements. Now we have an additive identity in our system of integers.

ADDING NEGATIVE INTEGERS

How shall we define the sum ${}^{-}m + {}^{-}n$, where m and n are any natural numbers? We shall wish to do it so that addition is commutative and associative, if that is possible.

1.4 Let's Explore

Let's just *suppose* that we can add any integers, and that the operation is commutative and associative.

1. Then think about ‘$(^-3 + {}^-2) + (3 + 2)$’. Would this name the same number as the symbol
   ‘$^-3 + (^-2 + 3) + 2$’? Why?
2. Would this be another name for the number
   $^-3 + (3 + {}^-2) + 2$? Why?
3. What about ‘$(^-3 + 3) + (^-2 + 2)$’?
   Would it name the same number? Why?
4. Find the simplest name for
   $(^-3 + 3) + (^-2 + 2)$.
5. Then what is the simplest name for
   $(^-3 + {}^-2) + (3 + 2)$?
6. If the sum of two integers is the additive identity, then what do we know about them?
7. Write a number sentence which says that $(^-3 + {}^-2)$ is the additive inverse of $(3 + 2)$.
8. Write the simplest name for $^-(3 + 2)$.
9. If addition of integers is to be commutative and associative, what is the simplest name for $^-3 + {}^-2$?
10. What idea does this give you about how to define the sum of two negative integers?
11. Using this idea, how would you define these sums?
    a. $^-7 + {}^-4$
    b. $^-14 + {}^-3$
    c. $^-8 + {}^-8$
    d. $^-30 + {}^-40$
    e. $^-5 + {}^-13$
    f. $^-13 + {}^-7$
    g. $^-16 + {}^-14$
    h. $^-45 + {}^-35$

If we wish addition to be commutative and associative then how must we define it? We saw that $^-3 + {}^-2$ must be $^-(3 + 2)$, or $^-5$. How about adding other negative integers? We could try the same idea as before for another pair of negative integers. Remember, we *supposed* we could add negative integers and that addition was commutative and associative.

If we do this for another pair of negative integers, this will help. But how could we figure out what must happen for *any* pair of negative integers? We shall use variables which can represent *any* natural number.

1.5 Let's Explore

1. Let's *suppose* that we can add any pair of negative integers, and that the operation is commutative and associative. Think about '$(^{-}4 + {}^{-}7) + (4 + 7)$'.
   a. Show that '$(^{-}4 + 4) + (^{-}7 + 7)$' must name the same number.
   b. Show that the number named is zero.
   c. Show that $^{-}4 + {}^{-}7$ and $4 + 7$ are additive inverses of each other.
   d. Write an equation which says that $^{-}4 + {}^{-}7$ is the additive inverse of $4 + 7$.
   e. Then what is the sum $^{-}4 + {}^{-}7$? Write the simplest name for it.

2. Again let's *suppose* that we can add any pair of negative integers, and that the operation is commutative and associative.

   This time let's use the symbols 'm' and 'n' to represent any natural numbers. Then the symbols '^{-}m' and '^{-}n' will represent any negative integers.

   Go through the argument of exercise 1, but do it for any negative integers ^{-}m and ^{-}n.

   Write your conclusion like this:

   If addition of negative integers is to be defined so that it will be commutative and associative, then for any negative integers ^{-}m and ^{-}n,

   ____.

3. You just proved that *if* addition is to be commutative and associative, you must define addition of negative integers in a certain way. Suppose you do define it this way. Have you proved that addition *will be* commutative and associative?

You now have an idea about how to define addition of negative integers. In fact you proved how you must define it, *if* you want addition to be commutative and associative. Let's define addition of negative integers this way. We could say that to add two negative integers, we

1. Add their absolute values, and then

2. Take the additive inverse of the sum.

For example, to add $^{-}3$ and $^{-}5$, we take the absolute values, 3 and 5. Then we add them, to get 8. Then we take the additive inverse of 8 to get $^{-}8$. Do you notice that the sum of two negative integers will always be negative?

Let's state our definition for adding negative integers.

For any natural numbers m *and* n,

$$^{-}m + {}^{-}n = {}^{-}(m + n)$$

(*To add negative integers add the absolute values and then take the additive inverse of the sum.*)

EXERCISES

Add, writing the simplest name for your result.

1. $^{-}12 + {}^{-}2$
2. $^{-}20 + {}^{-}30$
3. $^{-}1 + {}^{-}14$
4. $^{-}6 + {}^{-}6$
5. $^{-}43 + {}^{-}10$
6. $^{-}75 + {}^{-}13$
7. $^{-}89 + {}^{-}37$
8. $^{-}142 + {}^{-}113$
9. $({}^{-}7 + {}^{-}6) + {}^{-}8$
10. $^{-}7 + ({}^{-}6 + {}^{-}8)$
11. $({}^{-}12 + {}^{-}6) + {}^{-}2$
12. $^{-}12 + ({}^{-}6 + {}^{-}2)$
13. $({}^{-}15 + {}^{-}12) + {}^{-}8$
14. $^{-}8 + ({}^{-}15 + {}^{-}12)$
15. $({}^{-}5 + 5) + |3|$
16. $|{}^{-}7 + {}^{-}4| + {}^{-}11$
17. $|{}^{-}8| + ({}^{-}4 + {}^{-}4)$
18. $({}^{-}7 + |{}^{-}7|) + {}^{-}17$
19. $|6 + ({}^{-}2 + {}^{-}4)| + {}^{-}4$
20. $^{-}9 + |11 + {}^{-}11|$

Do we have the commutative and associative properties for addition of negative integers? By looking at a few examples it looks as if we might. Have we *proved* that the properties hold? Actually we did not. We proved that *if* the properties were to hold, *then* we had to make our definition as we did.

Perhaps we should think about If ________ then ________ sentences.

Example:

If a person lives in Chicago, then he lives in Illinois.

Suppose we turn the sentence around, like this:

If a person lives in Illinois, then he lives in Chicago.

The first sentence is true. The second one is not true. We interchanged the *if* clause and the *then* clause.

Now let's think again about our definition of addition. We proved that *if the properties were to hold, then we had to make our definition as we did.*

What about this sentence?

If we make our definition as we did, then the properties hold.

It isn't necessarily true. We did not prove it. Therefore we don't know whether addition of negative integers is commutative and associative. We can try to prove it later. Now let's go on to complete our definition.

ADDING A POSITIVE AND A NEGATIVE INTEGER

To see how to define the sum of a positive and a negative integer, we can *suppose* the operation to be commutative and associative. Then by some experimenting we can get an idea about how to make the definition.

1.6 Let's Explore

Suppose that we can add a positive and a negative integer, and that the operation is associative.

1. a. Think about $5 + {}^{-}2$.
   b. "Separate" the 5 into two parts, $3 + 2$. Then $5 + {}^{-}2 = (\quad) + {}^{-}2$.
   c. Use the associative property: $(3 + 2) + {}^{-}2 = 3 + (\quad)$.
   d. What is the simplest name for $3 + (2 + {}^{-}2)$?
   e. Then '$5 + {}^{-}2$' must be a name for what integer?
2. Use the procedure of exercise 1. In that way find:
   a. $7 + {}^{-}5$
   b. $4 + {}^{-}3$
   c. $8 + {}^{-}2$
   d. $10 + {}^{-}8$
   e. $6 + {}^{-}1$
   f. $14 + {}^{-}4$
   g. $12 + {}^{-}8$
   h. $15 + {}^{-}5$
   i. $9 + {}^{-}6$
   j. $10 + {}^{-}7$
3. In all parts of exercise 2, the positive number had a larger absolute value than the negative one. According to this experience, how would you define addition for this case? Write a definition in your own words. (Use variables or other symbols if you wish.)

You have an idea how to make one part of the definition. It is for the case in which the positive number has the larger absolute value. How about the case in which the negative integer has the larger absolute value?

1.7 Let's Explore

Again, *suppose* that we can add a positive and a negative integer, and that the operation is associative.

1. a. Think about $2 + {}^{-}5$.
   b. "Separate" the ${}^{-}5$ into two parts, ${}^{-}2 + {}^{-}3$. Then $2 + {}^{-}5 = 2 + (\quad)$.
   c. Use the associative property: $2 + ({}^{-}2 + {}^{-}3) = (\quad) + {}^{-}3$.
   d. What is the simplest name for $(2 + {}^{-}2) + {}^{-}3$?
   e. Then '$2 + {}^{-}5$' must be a name for what integer?

2. Use the procedure of exercise 1. In that way find:
   a. $3 + {}^{-}7$
   b. $4 + {}^{-}8$
   c. $10 + {}^{-}15$
   d. $20 + {}^{-}30$
   e. $18 + {}^{-}23$
   f. $14 + {}^{-}21$
   g. $12 + {}^{-}20$
   h. $11 + {}^{-}22$

3. How must we define addition in this case if we want the commutative and associative properties to hold? Write a definition in your own words.

4. Suppose a positive and a negative integer have the same absolute value. How shall we define the sum?

Let us now agree on our definitions. We still do not know what the properties are, but we can investigate them later. Here is the definition of addition for a positive integer and a negative integer. It has two parts.

For any natural numbers $\boldsymbol{m}$ *and* $\boldsymbol{n}$*,*

1. If $\boldsymbol{m > n}$*,*

$\boldsymbol{m + {}^{-}n = m - n}$. *Also* $\boldsymbol{{}^{-}n + m = m - n}$

(The positive integer has the larger absolute value. Find the difference of the absolute values. The sum will be positive.)

2. If $\boldsymbol{m < n}$*,*

$\boldsymbol{m + {}^{-}n = {}^{-}(n - m)}$. *Also* $\boldsymbol{{}^{-}n + m = {}^{-}(n - m)}$

(The negative integer has the larger absolute value. Find the difference of the absolute values and take the additive inverse. The sum will be negative.)

Our definition is now complete. We had already defined addition for whole numbers. We then made up additive inverses. This gave us a definition of addition of a positive and negative integer when each had the same absolute value. We also defined addition for zero and any negative integer. The only case left was for a positive and a negative integer with different absolute values. We made two cases out of this—one where the positive integer had the larger absolute value, and one where the negative integer had the larger absolute value.

Let's list the complete definition.

For any natural numbers $\mathbf{m}$ *and* $\mathbf{n}$*,*

A. $\mathbf{m} + {}^{-}\mathbf{m} = 0$ *and* ${}^{-}\mathbf{m} + \mathbf{m} = 0$ *(additive inverses)*

B. ${}^{-}\mathbf{m} + 0 = {}^{-}\mathbf{m}$ *and* $0 + {}^{-}\mathbf{m} = {}^{-}\mathbf{m}$ *(adding zero and a negative integer)*

C. ${}^{-}\mathbf{m} + {}^{-}\mathbf{n} = {}^{-}(\mathbf{m} + \mathbf{n})$ *(adding negative integers)*

D. If $\mathbf{m} > \mathbf{n}$*,*
$\mathbf{m} + {}^{-}\mathbf{n} = \mathbf{m} - \mathbf{n}$ *and* ${}^{-}\mathbf{n} + \mathbf{m} = \mathbf{m} - \mathbf{n}$ *(adding a negative and a positive integer)*

E. If $\mathbf{m} < \mathbf{n}$*,*
$\mathbf{m} + {}^{-}\mathbf{n} = {}^{-}(\mathbf{n} - \mathbf{m})$ *and* ${}^{-}\mathbf{n} + \mathbf{m} = {}^{-}(\mathbf{n} - \mathbf{m})$ *(adding a negative and a positive integer)*

Is addition of integers an operation? In other words are results unique? We can see that this is the case. Is the set of integers closed under addition? In other words, is the sum of *any* two integers also an integer? We can see that the set is closed. Is the operation commutative? Associative? We have not proved either. Later we shall find that the operation is both commutative and associative. For now you may assume these properties.

EXERCISES

In Exercises 1–6, add, writing the simplest name for your result.

1. a. $8 + {}^{-}3$
   b. ${}^{-}4 + 10$
   c. ${}^{-}9 + 14$
   d. $73 + {}^{-}8$
   e. $55 + {}^{-}25$
   f. ${}^{-}12 + 42$
   g. ${}^{-}180 + 220$
   h. $471 + {}^{-}21$
   i. $99 + {}^{-}88$
   j. ${}^{-}121 + 200$
   k. ${}^{-}91 + 217$
   l. $444 + {}^{-}333$

2. a. $2 + {}^{-}3$
   b. ${}^{-}5 + 4$
   c. ${}^{-}8 + 5$
   d. $6 + {}^{-}10$

e. $11 + {}^{-}13$
f. ${}^{-}19 + 15$
g. ${}^{-}25 + 15$
h. $75 + {}^{-}81$
i. $41 + {}^{-}50$
j. ${}^{-}88 + 66$
k. $39 + {}^{-}40$
l. ${}^{-}117 + 99$

3. a. ${}^{-}3 + {}^{-}7$
b. ${}^{-}8 + {}^{-}3$
c. ${}^{-}14 + {}^{-}6$
d. ${}^{-}12 + {}^{-}7$
e. ${}^{-}3 + {}^{-}91$
f. ${}^{-}31 + {}^{-}79$
g. ${}^{-}14 + {}^{-}112$
h. ${}^{-}51 + {}^{-}18$

4. a. $3 + {}^{-}2 + 4$
b. ${}^{-}5 + {}^{-}2 + {}^{-}14$
c. ${}^{+}14 + {}^{-}15$
d. ${}^{+}75 + {}^{-}10 + {}^{-}5$
e. $8 + 2 + {}^{+}4$
f. ${}^{-}18 + {}^{+}14 + {}^{-}12$
g. ${}^{-}200\% + 14 + {}^{-}1$
h. ${}^{-}21 + {}^{+}500\% + {}^{-}1200\%$

5. Hint: These may be easier if you associate the addends cleverly.
a. $14 + {}^{-}3 + {}^{-}2 + 12 + {}^{+}16 + {}^{-}8 + {}^{-}7 + 15$
b. ${}^{-}17 + {}^{-}12 + 14 + {}^{-}19 + 43 + 6 + {}^{-}20$
c. ${}^{-}13 + {}^{-}5 + {}^{-}7 + {}^{+}14 + {}^{-}3 + 41 + 22 + {}^{-}12$
d. $12 + {}^{-}15 + {}^{-}55 + 20 + {}^{+}4 + {}^{-}15 + {}^{-}13 + 6$

6. a. ${}^{-}3 + |{}^{-}4|$
b. $|{}^{-}5| + |{}^{-}8| + 2$
c. $|3 + {}^{-}10| + |{}^{-}1200\%|$
d. $|4| + {}^{-}|5 + {}^{-}7|$
e. $|3| + |{}^{-}5|$
f. $|3 + {}^{-}5|$
g. $|{}^{-}3| + |{}^{-}7|$
h. $|{}^{-}3 + {}^{-}7|$
i. $|7| + |{}^{+}5|$
j. $|7 + {}^{+}5|$
k. $|{}^{-}8| + |{}^{+}12|$
l. $|{}^{-}8 + {}^{+}12|$

7. Which of these sentences about integers are true?
a. ${}^{+}3 + {}^{-}2 > {}^{-}5$
b. ${}^{-}3 + {}^{+}4 = 1$
c. $|{}^{-}2| + 7 < {}^{+}9$
d. ${}^{-}3 < |2| + {}^{-}|{}^{-}4|$
e. ${}^{+}4 = |{}^{-}2| + |{}^{+}2|$
f. ${}^{-}5 = 3 + |{}^{-}1| + |{}^{+}1|$
g. ${}^{+}5 \geq |{}^{-}3 + 2|$
h. $|7 + {}^{-}5| \leq |7| + |{}^{-}5|$

8. Which of these sentences about integers are true?
a. $|7 + {}^{-}3| \leq |7| + |{}^{-}3|$
b. $|{}^{-}14 + {}^{+}7| \leq |{}^{-}14| + |{}^{+}7|$
c. $|35 + 2| \leq |35| + |2|$
d. $|{}^{-}7 + {}^{-}5| \leq |{}^{-}7| + |{}^{-}5|$
e. $|17 + {}^{+}3| \leq |17| + |{}^{+}3|$
f. $|{}^{-}12 + {}^{-}5| \leq |{}^{-}12| + |{}^{-}5|$
g. $|{}^{+}15 + {}^{-}25| \leq |{}^{+}15| + |{}^{-}25|$
h. $|{}^{-}19 + {}^{+}9| \leq |{}^{-}19| + |{}^{+}9|$
i. $|7 + {}^{-}3| = |7| + |3|$
j. $|{}^{-}14 + 7| = |{}^{-}14| + |7|$
k. $|17 + {}^{-}3| = |17| + |{}^{-}3|$
l. $|{}^{-}19 + 9| = |{}^{-}19| + |9|$
m. $|35 + 2| = |35| + |2|$
n. $|{}^{-}7 + {}^{-}5| = |{}^{-}7| + |{}^{-}5|$

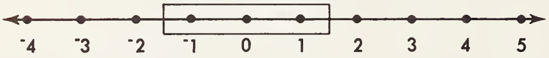
9. For each number sentence find the solution set in the system of integers.

   a. $\square + 6 = {}^{+}8$
   b. ${}^{-}7 = {}^{-}5 + \square$
   c. $3 + {}^{-}\square = {}^{-}5$
   d. ${}^{-}5 = {}^{-}\square + {}^{-}7$
   e. $x + 4 = {}^{-}2$
   f. $10 = 8 + {}^{-}x$
   g. $|x| + 2 = {}^{+}5$
   h. ${}^{-}4 = {}^{-}3 + |y|$
   i. $|t + 2| = 3$
   j. ${}^{+}6 = |{}^{-}s + {}^{-}4|$
   k. $|x| < 4$
   l. $|y| + 3 \leq 5$

10. Use a number line. Plot graphs of the sentences of exercise 9 g, i, j, k and l. Remember, to plot a graph of a number sentence, we draw a picture of its solution set on a number line.

Example:

$|x| < 2$
The solution set is $\{{}^{-}1, 0, 1\}$.
The graph is



*PROPERTIES OF ADDITION

We made up, or *defined*, addition for integers. We had to do it as we did in order to have the commutative and associative properties. We have still not proved that these properties hold. Let's do it now.

1.8 Let's Explore

1. Copy the definitions on p. 18. These tell how to add integers. There is one more case not listed. That is the case for which the integers are both whole numbers.

2. Look at the definitions and see whether addition is an operation. (Test every case, to see whether the result of adding is unique.)

3. Test for closure. (Check every case, to see whether you always get an integer when you add two integers.)

4. Is there an additive identity? What is it?

5. Does every integer have an additive inverse? How do you know?

6. a. What does the definition in A tell about the commutative property?
   b. What do the definitions in D and E tell about the commutative property?
7. Look at the definition in C. What does it tell us about the sum $^-n + {}^-m$?
8. What kind of numbers are represented by the variables m and n?
9. Is it true, then, that $m + n = n + m$?
10. Then is it true that $^-n + {}^-m = {}^-(m + n)$?
11. Then is it true that $^-n + {}^-m = {}^-m + {}^-n$ for any natural numbers m and n?

We have checked and found that addition is an operation. Also that there is closure. There is an additive identity and every integer has an additive inverse. The commutative property of addition holds. We still need to prove that addition is associative.

To test for the associative property we need to use our definitions. We can again check several cases. In each case, we test whether $x + (y + z) = (x + y) + z$. Thus there are three integers concerned in each case. The cases are:

1. All three integers positive or zero
2. All three integers negative
3. Two positive and one negative
4. One positive and two negative
5. One integer zero
6. Two integers zero

1.9 Let's Explore

1. Think about case 1. If all three integers are positive or zero, they are also what kind of numbers?
2. Is addition associative in this case?
3. Let's take the case of three negative integers. We want to know whether $^-m + ({}^-n + {}^-p) = ({}^-m + {}^-n) + {}^-p$, for all natural numbers m, n and p.
   a. Let's find another name for $^-m + ({}^-n + {}^-p)$. Use definition C on p. 18. Find $^-n + {}^-p$.
   b. Use the definition again. Find the sum of ^-m and your answer to a.
   c. Let's find another name for $({}^-m + {}^-n) + {}^-p$.
   Use the definition C again. Find $^-m + {}^-n$.

  d. Use the definition again. Find the sum of ^{-}p and your answer to c.
  e. Is it true that $^{-}m + (^{-}n + ^{-}p) = (^{-}m + ^{-}n) + ^{-}p$?

4. Now let's take the case in which one integer is negative and two are positive. We want to know whether $^{-}m + (n + p) = (^{-}m + n) + p$, for all natural numbers m, n and p.
  a. How shall we find $^{-}m + n$? Do you see that we need to know which has the larger absolute value? Let's assume, or suppose, that $|n| > |^{-}m|$. Then find $^{-}m + n$.
  b. Find $(^{-}m + n) + p$.
  c. Find $^{-}m + (n + p)$.
  d. Is addition associative in this case, provided $|n| > |^{-}m|$?
  e. Suppose that $|n| < |^{-}m|$, and that $|^{-}m| < |n + p|$. Check associativity under these conditions.
  f. Are there still other conditions we should try before we are sure about case 3?

EXERCISES

1. Investigate the associative property of addition for case 5 (one integer zero). Hint: Use variables 'x', 'y' which can represent any integer, positive or negative. You will not need additive inverse symbols.

2. Investigate the associative property of addition for case 6 (two integers zero).

*3. Investigate the associative property of addition for case 4 (two integers negative and one positive).

Review Practice

1. Rename each of these numbers without using the per cent sign.
  a. 100%
  b. 200%
  c. 50%
  d. 75%
  e. 10%
  f. 25%
  g. 20%
  h. $33\frac{1}{3}\%$
  i. $12\frac{1}{2}\%$
  j. $66\frac{2}{3}\%$
  k. $\frac{1}{2}\%$
  l. $.025\%$

2. Rename each of these numbers using a per cent sign.
  a. 1
  b. 5
  c. .5
  d. $\frac{3}{4}$
  e. $\frac{2}{10}$
  f. .73
  g. 1.25
  h. $\frac{1}{2}$

i. $\frac{1}{3}$
j. .01
k. $\frac{2}{3}$
l. $\frac{3}{5}$
m. .001
n. .00003

3. Find:
   a. 6% of 40
   b. 15% of 200
   c. 25% of 84
   d. 150% of 16
   e. 13% of 200
   f. 17% of 55
   g. 7% of 19
   h. .6% of 80

4. Find these square roots in the number system of arithmetic.
   a. $\sqrt{9}$
   b. $\sqrt{25}$
   c. $\sqrt{100}$
   d. $\sqrt{144}$
   e. $\sqrt{81}$
   f. $\sqrt{169}$
   g. $\sqrt{225}$
   h. $\sqrt{64}$
   i. $\sqrt{\frac{9}{16}}$
   j. $\sqrt{\frac{49}{100}}$
   k. $\sqrt{\frac{121}{36}}$
   l. $\sqrt{\frac{169}{225}}$

SUBTRACTION

When you studied clock arithmetic you learned about subtraction. It is the opposite of addition. In clock arithmetic every element has an additive inverse. We found we could subtract by adding an inverse. For example, in the arithmetic of the five minute clock

$$2 - 3 = 2 + {}^{-}3$$

In the system of integers all numbers have additive inverses, so we can subtract this same way. In fact, we can *define* subtraction to be the addition of the inverse.

For any integers x, y *the subtraction* $x - y$ *is defined to mean the same as the addition* $x + {}^{-}y$. (*To subtract* y *from* x, *we add the inverse of* y *to* x.)

Examples:

$$4 - 3 = 4 + {}^{-}3 = 1$$
$${}^{-}5 - 2 = {}^{-}5 + {}^{-}2 = {}^{-}7$$
$$7 - {}^{-}3 = 7 + {}^{+}3 = 10$$
$${}^{-}6 - {}^{-}8 = {}^{-}6 + {}^{+}8 = 2$$

EXERCISES

Subtract, writing the simplest name for your result.

1. $14 - 7$
2. $^{-}8 - 3$
3. $2 - 10$
4. $13 - {}^{-}7$
5. $2 - {}^{-}5$
6. $^{-}20 - 4$
7. $^{-}18 - {}^{-}7$
8. $^{-}4 - {}^{-}9$
9. $^{-}12 - {}^{-}6$
10. $17 - {}^{+}9$
11. $^{+}11 - {}^{+}15$
12. $^{-}6 - {}^{+}23$
13. $^{+}13 - {}^{-}16$
14. $^{-}15 - 2$
15. $12 - 3$
16. $^{-}5 - 20$
17. $^{-}18 - {}^{-}8$
18. $^{-}13 - 13$
19. $^{-}20 - {}^{-}20$
20. $^{-}24 - {}^{-}16$

Simplify (find the simplest name for each of these integers):

21. $|3|$
22. $|{}^{-}5|$
23. $|0|$
24. $|5 - 3|$
25. $|3 - 5|$
26. $|17 - 4|$
27. $|4 - 17|$
28. $|8 - {}^{-}3|$
29. $|{}^{-}15 - 2|$
30. $|{}^{-}5 - 20|$
31. $|21 + {}^{-}22|$
32. $|{}^{-}47 + 42|$
33. $^{-}({}^{-}14)$
34. $^{-}({}^{-}27)$
35. $^{-}|5|$
36. $^{-}|{}^{-}5|$
37. $^{-}|{}^{-}6|$
38. $^{-}|{}^{-}9|$
39. $^{-}|{}^{-}5 - 20|$
40. $^{-}|{}^{-}2 - 12|$
41. $^{-}5 - |7 - {}^{-}4|$
42. $7 + {}^{-}|{}^{-}6 + 5|$
43. $|{}^{-}14 - 1| - {}^{-}|{}^{-}6 - 22|$
44. $|{}^{-}17 + 5| + {}^{-}|14 - 21|$
45. $({}^{-}13 + 3) - {}^{-}5$
46. $({}^{-}3 - 6) - (4 - {}^{-}6)$
47. $(7 - 9) - {}^{-}({}^{-}3 - 5)$
48. $|(5 - 7) - {}^{-}({}^{-}3 - 5)|$

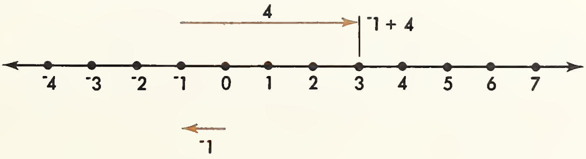
ADDITION AND SUBTRACTION ON THE NUMBER LINE

We have made up addition for integers. We did it in such a way that it would be commutative and associative. Now let's see how addition looks when we think of it on the number line.

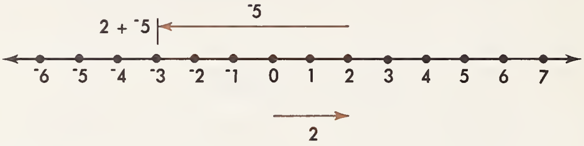
1.10 Let's Explore

1. Draw a number line for integers. Draw it positive to the right.
2. From the origin, which direction must you move your pencil on the line to get to each of these?
   a. $^{+}5$ b. 4 c. 7 d. $^{+}8$
3. Which direction must you move your pencil to get from the origin to each of these?
   a. $^{-}3$ b. $^{-}8$ c. $^{-}2$ d. $^{-}7$
4. Find the point 2. How far and in what direction must you move from 2 to find the point $2 + 5$?
5. Find the point $^{-}3$. How far and in what direction must you move from $^{-}3$ to find the point $^{-}3 + 5$?
6. Find the point $^{-}8$. How far and in what direction must you move from $^{-}8$ to find the point $^{-}8 + 3$?
7. Adding a positive integer can be represented by what kind of a move along the number line?
8. By what kind of move might addition of a negative integer be represented? Show how it works in these examples.
   a. $6 + {}^{-}4$ b. $5 + {}^{-}7$ c. $^{-}2 + {}^{-}3$

We have seen that adding a positive integer (or subtracting a negative integer) can be represented by a move to the right on a number line in standard position (with positive direction to the right). For example, $^{-}1 + 4$ is represented as follows:



Adding a negative integer (or subtracting a positive integer) can be represented by a move to the left. For example, $2 + {}^{-}5$ is represented like this:



EXERCISES

Add, using a number line.

1. ${}^{-}2 + 5$
2. $4 + 2$
3. ${}^{-}5 + 7$
4. $0 + 3$
5. ${}^{-}6 + 2$
6. ${}^{-}5 + 5$
7. $4 + {}^{-}1$
8. $7 + {}^{-}10$
9. ${}^{-}5 + {}^{-}2$
10. ${}^{-}2 + {}^{-}3$
11. $0 + {}^{-}4$
12. $4 + {}^{-}4$

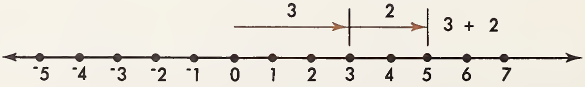
PROPERTIES OF ADDITION AND THE NUMBER LINE

We made up, or *defined*, addition for integers. We had to do it as we did in order to have the commutative and associative properties. If you did not study the optional section on page 20 you have not yet proved that these properties hold. You have only assumed that they hold, or taken the authors' or teacher's word for it.

We can use the number line idea to make these properties seem more reasonable.

1.11 Let's Explore

1. a. Draw a number line.
   b. Show the addition $3 + 2$, using arrows, like this:



   c. Show the addition $2 + 3$ in the same way.
   d. Is $3 + 2 = 2 + 3$?

2. a. Draw a number line. Mark the origin, but no other points.
   b. Use an arrow about an inch long for an integer a. Use an arrow about half as long for an integer b. Assume that a and b are both positive. Show $a + b$ on the line.
   c. Use the same line as in part b. Show $b + a$. (Use a different color if it helps.)
   d. Is $a + b = b + a$?

3. a. Draw a number line as you did for exercise 2. Mark only the origin.
   b. This time suppose a and b are both negative. Show $a + b$ on the line.
   c. Show $b + a$ on the line.

In the above exercises you have experimented with addition and the number line. Does your experience seem to prove the commutative property? In a way you have "proved" it. Most mathematicians would say you haven't made a real proof, however. To give a real proof you should not use the number line. You should use only the definition of addition. Anyway you have probably found this experience convincing. Can we show the associative property on a number line?

1.12 Let's Explore

1. a. Draw a number line, and label points for integers.
   b. Show the addition $2 + 1$, using arrows above the line.
   c. Now add 3 to $(2 + 1)$ and show the addition with another arrow. You have now shown $(2 + 1) + 3$.
   d. Use an arrow below the line to show 2.
   e. Now add $(1 + 3)$ to 2 and show the addition with another arrow below the line.
   f. Is $(2 + 1) + 3 = 2 + (1 + 3)$?

2. a. Draw a number line and label only the origin.
   b. Use arrows of different length for 3 integers a, b and c. Assume all three are positive and show $a + (b + c)$.
   c. Show $(a + b) + c$.
   d. Is $a + (b + c) = (a + b) + c$?

3. a. Draw a number line as before.
   b. Assume b and c are positive and a negative. Show $a + (b + c)$.
   c. Show $(a + b) + c$.
   d. Is $a + (b + c) = (a + b) + c$?

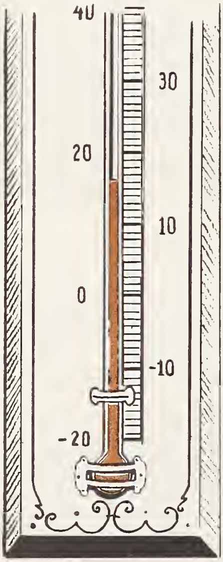
4. a. Draw a number line as before.
   b. Assume a and c are positive and b negative. Show $a + (b + c)$.
   c. Show $(a + b) + c$.
   d. Is $a + (b + c) = (a + b) + c$?
5. a. Draw another number line.
   b. Assume a and b are positive and c negative. Show $a + (b + c)$.
   c. Show $(a + b) + c$.
   d. Is $a + (b + c) = (a + b) + c$?

Does your experience in the above exercises show that addition of integers is associative? It probably seems most convincing. In a way you have "proved" it.

Remember that addition of integers is commutative and associative. Therefore the order and the grouping make no difference when computing a sum of integers.

THERMOMETERS

You have seen thermometers. Do you see that their scale is like a number line? Some of the numbers are less than zero. We can use our new number system—the system of integers—to work temperature problems. Here is an example:



> The temperature in Chicago at 10:00 a.m. was 17°. About noon a cold front moved past. The temperature dropped 30°. What was the temperature then?

Can you see that this is a subtraction problem with integers? We need to subtract 30 from 17. The answer is $^{-}13^{\circ}$, or 13° below zero.

Here is another example:

> The lowest temperature in Minneapolis on the night of December 12 was $^{-}31^{\circ}$. During the day the temperature rose 40°. What was the temperature then?

Can you see that this is an addition problem? We need to add 40 to $^{-}31$. The answer is 9°.

OTHER USES OF THE SYSTEM OF INTEGERS

When we made up clock arithmetic, we used the physical world for our ideas. We found that our arithmetic could also be used in other ways.

When we made up the system of integers we did not think about objects in the physical world. We invented negative integers so as to make a more complete number system. We made a system with commutative and associative addition. It also has an additive identity, and every number has an additive inverse.

We then found we could use this system to work thermometer problems. We shall also find other uses for it. Is this surprising? It shows that mathematical systems can be invented in two ways. From the physical world, or just by working with *ideas*. When we make them up just thinking about ideas, we often find that they are useful, too.

The system of integers could be used in card games. Here is an example of a problem:

> Mary and Bill are playing Pinochle. They are partners. Their opponents are Mary's mother and father. On the first hand Mary and Bill made a score of 210. Mother and father went set. They were 250 "in the hole." On the second hand Mary and Bill made 180 and mother and father made 210. On the third hand Mary and Bill went "in the hole" 150, and mother and father made 170. What were their total scores at this time?

This problem can be worked by adding integers, like this

	Mary & Bill	Mother & Father
1st hand	210	$^-250$
2nd hand	180	210
3rd hand	$^-150$	170
Total	240	130

We can use negative integers to represent debts. If a man has no money and owes somebody $100, we might use the integer $^{-}100$ to represent how much money he has. Here is a problem:

> Jerry Ward's father borrowed $350 from the bank last month. He also owes the finance company $125. He has $78 in the bank. Next week he gets his $300 paycheck. He will also get a $225 bonus check from his company. How much money will he then have?

This problem can also be worked using integers. We use negative integers for his debts and positive integers for the money which he has. The total is:

$$^{-}350 + {}^{-}125 + 78 + 300 + 225, \text{ or } 128$$

Therefore he will have $128 total.

EXERCISES

1. If the temperature is $^{-}40°$ F. and rises 32°, what will the temperature be?
2. Mercury freezes at $^{-}40°$ F. and water boils at 212° F. How much would the temperature rise in going from the first temperature to the second?
3. If the temperature during the day is 17° F. and it drops 25° during the night, how cold will it then be?
4. Michael and Nathan are partners in a card game. On four hands they make scores of 170, $^{-}200$, $^{-}50$, and 320, respectively. What is their total score?
5. Jim's father is a businessman. He made 5 transactions. On the first one he made a profit of $73. On the second he lost $42. On the others he made a profit of $12, a loss of $21 and a loss of $85. What was his total profit (or loss)?
6. Make up two problems (for which you can use integers) about temperatures.
7. Make up two problems about gains and losses in a football game.
8. Make up two problems about gains and losses in the stock market.
9. Add:

Example:

$$\begin{array}{r} ^{-}351 \\ 725 \\ 453 \\ ^{-}218 \\ ^{-}101 \\ \underline{387} \end{array}$$

It is usually better to add the positive numbers separately. Also add the negative numbers separately. (We can do this because of the commutative and associative properties.) Then add the two sums, like this.

$$\begin{array}{r} 725\\ 453\\ 387\\ \hline 1565 \end{array} \qquad \begin{array}{r} {}^-351\\ {}^-218\\ {}^-101\\ \hline {}^-670 \end{array} \qquad \begin{array}{r} 1565\\ {}^-670\\ \hline 895 \text{ ans.} \end{array}$$

a. $$\begin{array}{r} 15\\ {}^-104\\ 256\\ {}^-489\\ {}^-210\\ \hline \end{array}$$

b. $$\begin{array}{r} {}^-9847\\ 251\\ 8401\\ 111\\ {}^-2565\\ 312\\ \hline \end{array}$$

c. $$\begin{array}{r} 456\\ {}^-356\\ 722\\ 418\\ {}^-4516\\ 895\\ \hline \end{array}$$

d. $$\begin{array}{r} {}^-756\\ 812\\ 403\\ {}^-980\\ 1240\\ {}^-2120\\ \hline \end{array}$$

e. $$\begin{array}{r} 496\\ {}^-313\\ {}^-965\\ 2140\\ 1240\\ {}^-2682\\ \hline \end{array}$$

f. $$\begin{array}{r} {}^-9010\\ 8125\\ 4003\\ {}^-6525\\ 391\\ 3142\\ \hline \end{array}$$

g. $$\begin{array}{r} 4512\\ 802\\ {}^-6143\\ 1101\\ {}^-2023\\ 493\\ \hline \end{array}$$

h. $$\begin{array}{r} {}^-3514\\ 700\\ 2103\\ {}^-4770\\ 9412\\ {}^-1234\\ \hline \end{array}$$

i. $$\begin{array}{r} {}^-2103\\ {}^-4615\\ {}^-7142\\ {}^-8517\\ {}^-3841\\ {}^-1037\\ \hline \end{array}$$

NUMBER SENTENCES

Mathematics is a special kind of language. It is important for us to learn about its sentences and phrases.

We have used number sentences already. A number sentence is one which refers to numbers. The equations we have used are number sentences, and so are the inequalities.

We agreed on the meaning we would give to an equation. Do you remember? Every equation has symbols on the right and also on the left of the verb =. An equation says that the symbols on the right and on the left name the same number. For example, the equation $5 - {}^-6 = 9 + {}^-2$ says that '$5 - {}^-6$' and '$9 + {}^-2$' are numerals for the same number. Is the sentence true?

Equations may be true or false. Sometimes they are neither. Here is an equation which is neither.

$$x + {}^{-}3 = 5$$

In this sentence there is a variable, 'x'. We can replace it with various names of numbers. When we make a replacement we sometimes get a true sentence. We sometimes get a false sentence. The equation is neither true nor false as it stands.

Do you remember that every equation is reversible?

INEQUALITIES

A sentence with any of these verbs is called an *inequality*. Let's review the meaning of each.

$\neq$	is not equal to
$<$	is less than
$\leq$	is less than or equal to
$>$	is greater than
$\geq$	is greater than or equal to.

An inequality with $\neq$ for its verb says that the symbols on the right and left of the verb name different numbers. For example, the sentence $3 + {}^{-}2 \neq 5$ says that '$3 + {}^{-}2$' and '5' name different numbers.

An inequality like ${}^{-}3 < 5 + {}^{-}2$ says that the number named on the left is less than the number named on the right. This sentence is true. Some inequalities are false. When an inequality has variables it may be neither true nor false.

NEGATIONS

Compare these sentences.

$$5 - 2 = 4 + {}^{-}1$$
$$5 - 2 \neq 4 + {}^{-}1$$

They say opposite things. The first one is true. The second is false. They are called *negations* of each other. The second is the negation of the first. The first is the negation of the second.

How can we negate a sentence with $=$ for its verb? We simply change the verb to $\neq$.

For example, to negate

$$2 + {}^{-}x = {}^{-}3$$

we simply write

$$2 + {}^{-}x \neq {}^{-}3.$$

How can we negate a sentence with $\neq$ for its verb? We simply change the verb to $=$.
For example, to negate

$$^-25 - 4 \neq 17$$

we simply write

$$^-25 - 4 = 17.$$

1.13 Let's Explore

1. Negate each of these sentences. Tell which are true and which false.
   a. $1 + {}^-3 = 5$
   b. $3 - x = {}^-5$
   c. $4 + {}^-x - 7 \neq x + 2$
   d. $3 - |{}^-y| \neq y + 2$

2. Let's think about the sentence $x \leq 5$. If we replace 'x' by a name for zero, we get the sentence $0 \leq 5$. It is true.
   Replace 'x' by a name for:
   a. 2
   b. 3
   c. 4
   d. 5
   e. 6
   f. 7
   Which sentences are true? Which are false?

3. Let's think about the sentence $x > 5$. If we replace 'x' by a name for 9, we get the sentence $9 > 5$. It is true.
   Replace 'x' by a name for:
   a. 2
   b. 3
   c. 4
   d. 5
   e. 6
   f. 7
   Which sentences are true? Which are false?

4. Compare $x \leq 5$ and $x > 5$.

5. We can write $x \ngtr 5$ for the negation of $x > 5$. How else could you write a negation of $x > 5$?

To negate a sentence with the verb $>$ we can simply change the verb to $\ngtr$ (read "is not greater than"). You have probably discovered that we can also negate such a sentence by changing the verb to $\leq$.

To negate a sentence we can change the verb as follows:

$$<\ \longleftrightarrow\ \nless$$
$$\leq\ \longleftrightarrow\ \nleq$$
$$>\ \longleftrightarrow\ \ngtr$$
$$\geq\ \longleftrightarrow\ \ngeq$$
$$=\ \longleftrightarrow\ \neq$$

EXERCISES

1. Tell, in words, the meaning of each sentence.

Example:

The sentence $3 + 2 = 4 + 5$ says that '$3 + 2$' and '$4 + 5$' name the same number.

a. $3 + {}^-2 = 4 - {}^-1$
b. $5 - |{}^-2| < 4$
c. $4 + x \neq 9 - 2$
d. $6 + 5 \geq x - {}^-3$

2. To negate a sentence with $>$ for its verb we can change the verb to $\ngtr$. We can also change the verb to $\leq$. This way we do not use the symbol | as a mark through a verb.

Find how to negate sentences having these verbs without using the symbol | through a verb.

a. $<$
b. $\leq$
c. $\geq$

3. Negate each of these sentences.

a. $4 + 2 = 5 + {}^-7$
b. $5 - {}^-|x| \neq 4 + 2$
c. $3 - x < 12$
d. ${}^-5 \leq p - 5$
e. $9 + {}^-t > t$
f. ${}^-14 \geq 13 + y - {}^-4$

NUMBER PHRASES

Number phrases are symbols which can be used as certain parts of sentences. We have used them many times. For example, remember how we defined subtraction. We said that for all integers x and y, $x - y = x + {}^-y$. What did this mean? When we replace the variables on the left by names for any integers we get the name of some integer. Also when we make the same replacements on the right we get the name of an integer.

We would say that '$x - y$' is a number phrase. It is (or can be) part of a number sentence. When we make replacements for the variables we can get the name of a number.

Let's think of the sentence ${}^-3 + 2 = y$. In this case 'y' is a phrase. We can get the name of a number by replacing this variable with a numeral. The symbol '${}^-3 + 2$' is also a phrase. It is already the name of a number.

Phrases are also called *expressions*. They are made up of numerals, variables, plus signs, minus signs, parentheses, etc. For a symbol to be a phrase, or expression, it must be able to name a number upon replacements for the variables, or already be the name of a number.

Here are some examples of phrases:

‘$^{+}5$’, ‘$x + y$’, ‘$x + {}^{-}t - 2$’, ‘$p + {}^{-}3$’, ‘$^{-}x + y$’

Here are some symbols which are not number phrases because they can not name a number, for any replacement.

‘$x - ($’, ‘$^{+} - |p$’, ‘$\div 8$’, $\dfrac{\text{‘}x\text{’}}{0}$

For every phrase we should have in mind a replacement set. This is the set of numbers we use when we make replacements of the variables. We replace a variable by a name for some number in the replacement set. When we are speaking about the integers, our replacement set is the set of integers. In other cases the replacement set may be different.

A symbol (or set of symbols) is a ***number phrase*** *(or expression) if and only if it is*
1. A name of some number, or
2. Can name a number for some replacement of its variables.

EXERCISES

Which of these is a number phrase (or expression)? For each number phrase with a variable give a replacement which produces the name of some number.

Symbol	Replacement Set
1. x	The set of integers
2. $^{-}5 +)$	The set of integers
3. $x + {}^{-}y$	The set of integers
4. $2 \cdot (a + b)$	The set of numbers of arithmetic
5. $x \cdot \left(2 - \frac{3}{x}\right)$	The set of whole numbers
6. $\frac{4}{x}$	The set of natural numbers
7. $\frac{4}{x}$	The set of whole numbers

8. $\frac{4}{x}$	The set of integers
9. $\frac{4}{x}$	The set of arithmetic numbers
10. $\frac{p}{0}$	The set of natural numbers
11. $\frac{3}{x - x}$	The set of numbers of arithmetic
12. $3 + {}^{-}2$	The set of integers
13. ${}^{-}x$	The set of integers
14. ${}^{-}({}^{-}x)$	The set of integers
15. ${}^{-}(a + b)$	The set of integers
16. ${}^{-}a + {}^{-}b$	The set of integers
17. $\lvert x\rvert + \lvert y\rvert$	The set of integers
18. $\lvert x + y\rvert$	The set of integers

Review Practice

1. Write exponential notation for each of the following:
   a. $3 \cdot 3 \cdot 3 \cdot 3$
   b. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
   c. $10 \cdot 10$
   d. $10 \cdot 10 \cdot 10 \cdot 10$
   e. $7 \cdot 7 \cdot 7$
   f. $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$

2. Factor each of these natural numbers into primes. Use exponential notation in writing your answer.
   a. 76
   b. 288
   c. 1000
   d. 264
   e. 64
   f. 81
   g. 160
   h. 98

3. Find the L.C.M. of each set of numbers. (Remember to factor first.)
   a. 12 and 15
   b. 16 and 18
   c. 7 and 21
   d. 10 and 25
   e. 12, 18 and 21
   f. 3, 12 and 16

4. Add: (Remember to find the L.C.M. of the denominators. Then multiply by 1 where necessary to make all denominators the same.)
   a. $\frac{3}{9} + \frac{2}{9}$
   b. $\frac{4}{17} + \frac{12}{17}$

c. $\frac{2}{3} + \frac{5}{6}$

d. $\frac{7}{15} + \frac{2}{5}$

e. $\frac{4}{7} + \frac{4}{21}$

f. $\frac{2}{3} + \frac{5}{6} + \frac{2}{9}$

g. $\frac{9}{10} + \frac{3}{5} + \frac{4}{25}$

h. $\frac{8}{3} + \frac{1}{12} + \frac{3}{16}$

5. Subtract:

a. $\frac{12}{5} - \frac{4}{5}$ b. $\frac{11}{12} - \frac{3}{4}$ c. $\frac{7}{9} - \frac{1}{6}$ d. $\frac{3}{8} - \frac{1}{12}$

6. Show that each of these numbers is an arithmetic number. (Show that it can be named with a fractional numeral with a whole number for numerator and a natural number for denominator.)

Examples:

$$1.25 = \frac{125}{100} \qquad 5 = \frac{5}{1}$$

$$\frac{1}{\frac{2}{3}} = \frac{1}{\frac{2}{3}} \cdot \frac{\frac{3}{2}}{\frac{3}{2}} = \frac{\frac{3}{2}}{1} = \frac{3}{2}$$

a. 3.5

b. .06

c. $\frac{2}{\frac{4}{5}}$

d. $\frac{\frac{1}{2}}{5}$

e. 16%

f. 5^3

EQUIVALENT PHRASES

Sometimes we find that two different number phrases are very much alike.

1.14 Let's Explore

1. We wish to compare two phrases, '$x - 2$' and '$^{-}2 + x$'. To do this let's make various replacements for 'x' using the set of integers as the replacement set. Keep your results in a table, like this:

x	$x - 2$	$^{-}2 + x$
2	0	0
4		
$^{-}5$		
$^{-}27$		
0		
12		
35		
126		
$^{-}425$		

When we replace 'x' by '2' we get '$2 - 2$', or '0' for the first phrase.

For the second phrase we get '$^{-}2 + 2$', or '0'. These results are written in the table. You complete the table.

2. What relation do you see between the phrases ‘$x - 2$’ and ‘$^{-}2 + x$’?

3. Compare the phrases ‘$^{-}(x + y)$’ and ‘$^{-}x + {}^{-}y$’. Again keep your results in a table

x	y	$^{-}(x + y)$	$^{-}x + {}^{-}y$
$^{-}3$	2	1	1
4	$^{-}5$		
6	3		
$^{-}8$	$^{-}5$		
$^{-}6$	$^{-}3$		
0	4		
$^{-}12$	0		

Replacing ‘x’ by ‘$^{-}3$’ and ‘y’ by ‘2’, we get ‘$^{-}(^{-}3 + 2)$’, or ‘$^{-}(^{-}1)$’, or ‘1’, for the first phrase, and ‘$^{-}(^{-}3 + {}^{-}2)$’, or ‘$3 + {}^{-}2$’, or ‘1’, for the second phrase. You complete the table.

4. What relation do you see between the phrases ‘$^{-}(x + y)$’ and ‘$^{-}x + {}^{-}y$’?

We have seen something important. Sometimes two phrases name the same number for *all* numbers in the replacement set. You did not try all replacements. But did you see that this is probably true in the above examples? Later we shall learn to prove when two phrases have this property.

Phrases, or expressions, like these are called *equivalent*. We say two expressions are *equivalent to each other*, or we can simply say they are *equivalent*.

Two number phrases are ***equivalent*** *if they name the same number for all numbers in the replacement set.*

We have not really *proved* phrases to be equivalent. It is easier to prove phrases are not equivalent. We look for a replacement for which they do not name the same number. If we can find one, we have proved they are not equivalent.

Example:

Prove that the phrases ‘$^{-}(y + 3)$’ and ‘$^{-}y + 3$’ are not equivalent.

Replacing ‘y’ by ‘3’ we get ‘$^{-}(3 + 3)$’, or ‘$^{-}6$’, for the first phrase. For the second phrase we get ‘$^{-}3 + 3$’, or ‘0’.

EXERCISES

Tell which of these are equivalent pairs of phrases. For the equivalent ones, make a table for at least five replacements from the set of integers. For those not equivalent, prove that they are not, by an example.

1. $a + b; b + a$
2. $a - b; a + {}^{-}b$
3. $x - y; y - x$
4. ${}^{-}({}^{-}x); x$
5. $2 - {}^{-}x; x + 2$
6. $3 - {}^{-}p; 3 - p$
7. ${}^{-}(x - 3); {}^{-}x + 3$
8. $x - y; {}^{-}(y - x)$
9. $5 + (b + {}^{-}c); (5 + b) - c$
10. $3 - (x - 4); (3 - x) + 4$
11. $a - (b - c); (a - b) - c$
12. $4 - (5 - y); 4 + {}^{-}5 + y$
13. $p + (a + {}^{-}a); (p + a) + {}^{-}a$
14. $5 + {}^{-}2; 5 - 2$
15. ${}^{-}13; {}^{-}10 + {}^{-}3$
16. $|x|; |{}^{-}x|$
17. $|x - y|; |y - x|$
18. ${}^{-}|x|; {}^{-}x$

SIMPLIFYING PHRASES

You have worked many simplification problems. Remember, the directions for some problems say: "Find the simplest name for this number." For example, the simplest name for ${}^{-}13 + 15$ is '2'.

We found a phrase *equivalent* to '${}^{-}13 + 15$'. The new one we found was simpler. This is what is meant by *simplifying* a phrase.

Example:

Simplify: ${}^{-}({}^{-}x)$.

The simplest phrase we can find equivalent to '${}^{-}({}^{-}x)$' is the phrase 'x'. So 'x' is the answer.

EXERCISES

Simplify:

1. $12 - {}^{-}2$
2. $4 + (x - x)$
3. $(y - x) + {}^{-}y$
4. $|{}^{-}x|$
5. $3 - (x - 4)$
6. $4 + (5 - y)$
7. ${}^{-}({}^{-}(x - 2))$
8. ${}^{-}y + x$

SIMPLIFYING PHRASES TO SOLVE EQUATIONS

Let's think about the equation

$$3 - (x - 4) = 2.$$

To solve it, we must find the set of numbers that make it true. We can simplify the left side. We already did, in exercise 5 above. We know that '$3 - (x - 4)$' simplifies to '$7 - x$'. These two phrases are equivalent. This means they name the same number for *all* replacements.

Let's replace the left side of the equation by '$7 - x$'. Then we have

$$7 - x = 2.$$

Any replacement making this equation true will also make the first one true. Any replacement making this one false will also make the first one false. The two equations must have the same solution set.

Which of the two equations is easier to solve? Do you see how simplifying phrases can make it easier to solve equations? We replace a phrase by an equivalent, simpler one. Then we solve the simpler one.

Example:

> Find an equation simpler than $2 = 4 + (5 - y)$, but with the same solution set.

We know that '$9 - y$' is equivalent to '$4 + (5 - y)$'. So the simpler equation is $2 = 9 - y$.

EXERCISES

For each sentence, find one which is simpler, but having the same solution set.

1. $(y - 2) + {}^{-}y = y + 1$
2. $x - 2 = (x + 4) - x$
3. ${}^{-}(2 + {}^{-}x) + 1 = 3$
4. $4 + {}^{-}(x + 2) = 5$
5. ${}^{-}(2 - {}^{-}x) = 3$

CONDITIONAL SENTENCES

A certain kind of sentence is called a *conditional sentence.* It is important in mathematical reasoning and other kinds of reasoning. Here are some examples.

1. If an integer is larger than 5, then it is positive.
2. If a number is negative, then it is less than 2.
3. If you wish to subtract, then you may add the inverse.

4. If $x = 2$, then $x + 2 = 4$.
5. If you sell your quota, then you will win a prize.
6. If $x < 7$, then $x < 5$.
7. If a number is larger than 5, then it is negative.

Notice that each sentence has two parts, or *clauses*. One of them follows the word *if*. The other follows the word *then*. The clause following ‘if’ is called the *antecedent*, and the clause following ‘then’ is called the *consequent*.

1.15 Let’s Explore

1. For each of the seven sentences above, pick out the antecedent and the consequent.
2. Make up 5 conditional sentences. Pick out the antecedent and the consequent of each.

Some conditional sentences are true. Some are false. Some are neither true nor false.

1.16 Let’s Explore

1. Look at sentences 1–7 above. Tell which seem to be true. Which seem to be false? Which seem to be neither true nor false?
2. The following sentence seems to be true. Its replacement set is the set of integers.

$$\text{If } x > 5, \text{ then } x > 0.$$

If we replace ‘x’ by ‘6’ the antecedent becomes true. The consequent also becomes true.

Make these replacements for ‘x’. Keep track of the truth of antecedent and consequent. Keep your results in a table, like this. Mark ‘T’ for true, and ‘F’ for false.

x	antecedent ($x > 5$)	consequent ($x > 0$)
6	T	T
25		
$^{-}5$		
$^{-}12$		
0		
3		

3. What is the solution set of $x > 5$?
4. What is the solution set of $x > 0$?
5. Could you possibly find a replacement making the antecedent true and the consequent false?
6. Is the solution set of $x > 5$ a subset of the solution set of $x > 0$?
7. Is the solution set of $x > 0$ a subset of the solution set of $x > 5$?

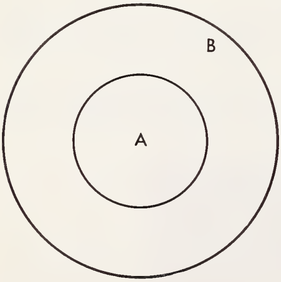
The sentence we have been studying is surely true. It says that if an integer is larger than 5, then it must be larger than 0. When we made replacements for the variable, the antecedent sometimes became true. It sometimes became false. The consequent also was sometimes true and sometimes false. But *never* could we make the antecedent true and the consequent false with the same replacement. This is really what a conditional sentence means.

Let's use variables to represent sentences. It will be easier to talk about them that way. We can talk about sentences like "If A, then B." Here the variable 'A' represents the antecedent and 'B' the consequent. We can also symbolize this kind of sentence more simply, like this:

$$A \rightarrow B$$

When we write the arrow, it takes the place of the words *if* and *then*. We may read $A \rightarrow B$ as "if A then B" or, "A implies B."

Perhaps you noticed that for a conditional sentence $A \rightarrow B$ to be true the solution set of the antecedent must be a subset of the solution set of the consequent, like this:



This Venn diagram shows the solution sets of A and B. Any replacement which makes A true is also in the solution set of B. So it makes B true also. Using this diagram, can you show that A might be false, while B is true? Can you show that A and B might both be false?

A table might also help show when a conditional sentence $A \rightarrow B$ is true.

A	B	$A \rightarrow B$
T	T	T
F	T	T
F	F	T
T	F	F

If $A \rightarrow B$ is true then the antecedent, A, cannot be true while the consequent, B, is false.

The important thing to remember is the following:

> *A* ***conditional sentence,*** ***A*** $\rightarrow$ ***B,*** *is true when:*
> *There is no replacement making* ***A*** *true and* ***B*** *false.*
> *The sentence is false when:*
> *There is at least one replacement making* ***A*** *true and* ***B*** *false.*

EXERCISES

1. In each sentence pick out the antecedent and consequent.
   - a. If Mary is mad, then Joe is glad.
   - b. If $2 < 3$, then $5 = 5$.
   - c. If $2 + x = 2 - x$, then $x = 0$.
   - d. If $x = y$, then $|x| = |y|$.
   - e. If $|x| = |y|$, then $x = y$.
   - f. If $p \leq 2$, then $p \neq 3$.
   - g. If $x + y = 0$, then $x = {}^{-}y$.
   - h. If $x = {}^{-}y$, then $x + y = 0$.
   - i. $y = 2 \rightarrow y < 5$
   - j. $y < 5 \rightarrow y = 2$
2. The replacement set for each sentence is the set of integers. Tell which sentences are true, which are false. Prove the false ones to be false, by finding a replacement making the antecedent true and the consequent false.
   - a. If $y = {}^{-}4$, then $y \leq {}^{-}2$.
   - b. If $a = b$, then $|a| = |b|$.
   - c. If $x = {}^{-}y$, then $x + y = 0$.
   - d. If $t \geq {}^{-}6$, then $t > {}^{-}8$.
   - e. If $p \leq 10$, then $p < 5$.
   - f. If $r < |x|$, then $r < x$.
   - g. If $r < x$, then $r < |x|$.
   - h. If $y = x - 2$, then $y = x + {}^{-}2$.
   - i. If $x = |a + b|$, then $x = |a| + |b|$.
   - j. If $|a| = |b|$, then $a = b$.

CONVERSES

Have you noticed that a conditional sentence can be turned around? In the previous exercises, some of the sentences were alike except for one thing. The antecedent and consequent were interchanged.

Examples:

$A \rightarrow B$	$B \rightarrow A$
If $y < 5$, then $y < 10$.	If $y < 10$, then $y < 5$.
If $a = b$, then $\lvert a\rvert = \lvert b\rvert$.	If $\lvert a\rvert = \lvert b\rvert$, then $a = b$.

How are such pairs of sentences related?

1.17 Let's Explore

1. Study this conditional sentence:

$$y < 5 \rightarrow y < 10$$

Is it true or false?

2. Interchange the antecedent and consequent. Is your new sentence true or false?

3. Study this conditional sentence:

$$x = {}^{-}y \rightarrow x + y = 0$$

Is it true or false? How do you know?

4. Interchange the antecedent and consequent. Is your new sentence true or false?

5. Study this conditional sentence:

$$x = 1 \rightarrow x = 2$$

Is it true or false?

6. Interchange the antecedent and consequent. Is your new sentence true or false?

When we interchange antecedent and consequent we get a new sentence. It is called the *converse* of the first sentence. You have discovered that a sentence might be true, but its converse false. It may be true and its converse also true. It may be false and its converse also false. This is important. People make mistakes in reasoning by forgetting this fact.

Remember how we made up the integers? We said *if* we want the commutative and associative properties, *then* we must define addition as we did. What is the converse? *If* we make the definition as we did, *then* addition will be commutative and associative. The converse is not necessarily true. We had to check it to see.

The ***converse*** *of a conditional sentence* $A \rightarrow B$ *is the sentence* $B \rightarrow A$*. The converse is obtained by interchanging antecedent and consequent. The converse of a true sentence might not be true.*

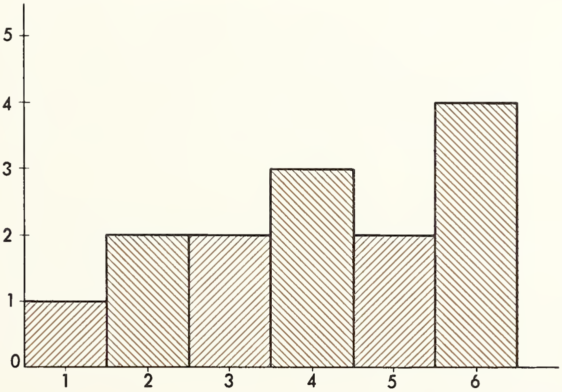
EXERCISES

1. Write the converse of each sentence.
   - a. If $p < {}^{-}1$, then $p < 0$.
   - b. If $y = {}^{-}3$, then $y + 2 = {}^{-}1$.
   - c. If $2 < 3$, then $5 = 5$.
   - d. $x = y \rightarrow |x| = |y|$
   - e. If $p \leq 2$, then $p \neq 3$.
   - f. $t \geq {}^{-}6 \rightarrow t > {}^{-}8$
2. In exercise 1, which sentences are true, and which false?
3. In exercise 1, which converses are true, and which false?
4. In exercise 1, which true sentences have true converses?

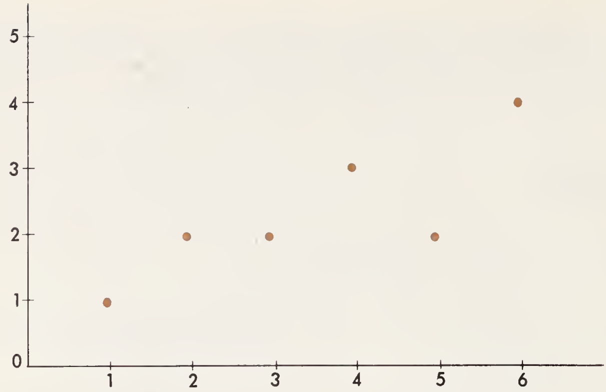
GRAPHS ON A PLANE

We have already studied graphs of number sentences. So far, the sentences we graphed had only one variable. We graphed them on a number line. We can also make graphs on a plane.

Do you remember making bar graphs? Here is an example:



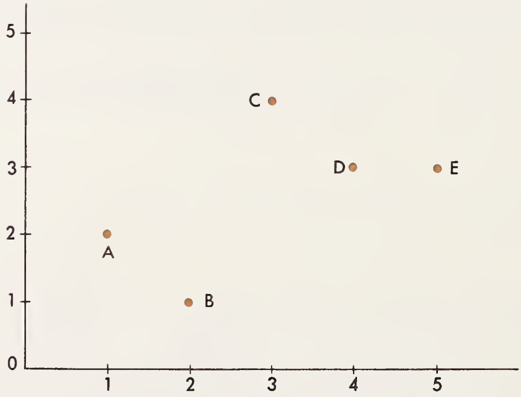
This is a graph on a plane. We could have drawn it using dots, as on the next page.



How do you tell where to put the dots?

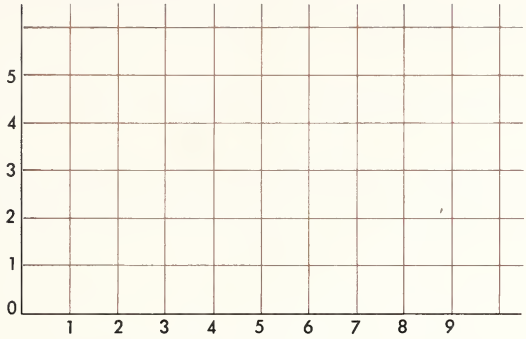
1.18 Let's Explore

Here is a graph on a plane.



1. Look at point A. How far to the right is it from the edge of the graph? How far up from the bottom of the graph is it?
2. Look at point C. How far to the right is it?
   How far up is it?
3. By telling how far they are to the right and how far up, give the locations of points B, D and E.

4. Let's make a graph. Make a drawing like this. (If you use graph paper it will be easier.) Number it as shown.



5. Make a dot 3 units to the right and 2 units up. Label it 'A'.

6. Make a dot 2 units to the right and 4 units up. Label it '(2, 4)'.

7. Make a dot 5 units to the right and 3 units up. Label it '(5, 3)'.

8. Make a dot 3 units to the right and 5 units up. Label it '(3, 5)'.

9. Make a dot 0 units to the right and 2 units up. Label it '(0, 2)'.

10. Make a dot 2 units to the right and 0 units up. Label it '(2, 0)'.

For graphs on a line we can find a point when we know one number. For graphs on a plane we need a *pair* of numbers to locate a point. The numbers of this pair must be in the right order. One of them tells how far a point is to the right, and the other tells how far up it is. We must know which is which. We agree to mention the distance to the right first.

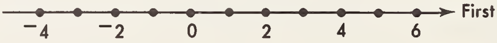
We wrote symbols like '(3, 2)' to label points. These are symbols for *ordered pairs* of numbers. Names of ordered pairs of numbers were also names of points on our graph.

INTEGERS AND GRAPHS

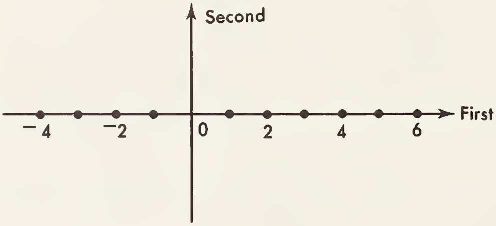
We have drawn graphs on a line, using whole numbers. We have also used the numbers of arithmetic and the integers. We have seen how to draw graphs on a plane. Let's use the system of integers now, with graphs on a plane.

1.19 Let's Explore

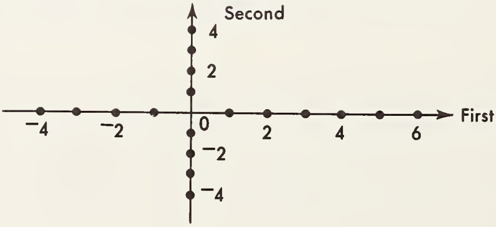
1. Draw and label an integer number line. Use graph paper. Draw the line in the middle of the paper, extending to the right. Label the positive half-line "First," like this.



2. Find the zero point. Through this point, draw another line. Make it perpendicular to the first one, in the middle of the paper, like this.



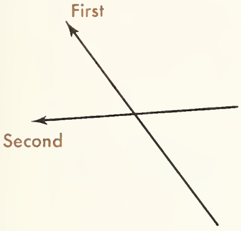
3. Make this new line an integer number line. Put the points for positive integers above the first line. The points for the negative integers will be below the line. Label the positive half-line "Second," like this.



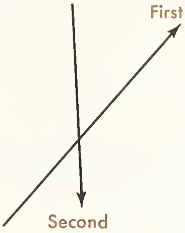
4. Draw a dot at (5, 1). Label it '(5, 1)'.
5. Draw and label a dot at (⁻3, 2).
6. Draw and label a dot at (⁻2, ⁻4).
7. Draw and label a dot at (5, ⁻4).
8. Draw and label a dot at (⁻4, 0).
9. Draw and label a dot at (0, 0).

We have learned how to use integers to make graphs on a plane. Let's review. We first draw two integer number lines. These are called *axes*. One of them is called the *first axis*. The other is called the *second axis*.

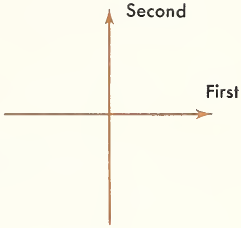
There are many ways to draw pairs of axes. Here are some.



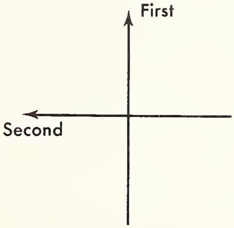
a.



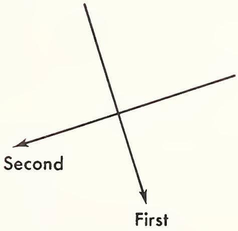
b.



c.



d.

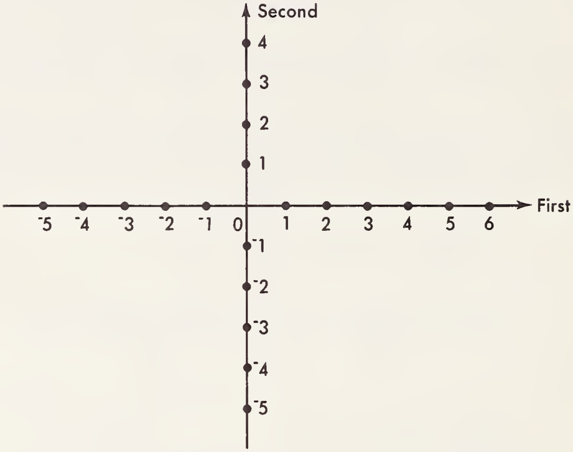


e.

The arrows show the positive direction on an axis. Notice that some of the pairs of axes are perpendicular. Others are not. The first axis could point up, down, right, or left. So could the second axis. Scientists often use axes in positions like these. It sometimes makes their work simpler. To study graphs in mathematics we have a *standard position* for axes. It is like (c) above. The first axis points to the right. The second axis points up. The axes are perpendicular. From now on let's all draw axes in standard position unless we need them another way.

We found that an *ordered pair* of integers identifies a point on the plane. We name an ordered pair like this ‘($^-3$, 4)’. The first member of this pair is $^-3$. The second member is 4. The numbers of an ordered pair are also called the *coordinates* of the point. We say that $^-3$ is the *first coordinate* and that 4 is the *second coordinate*. The first coordinate tells how far a point is from the second axis. The second coordinate tells how far a point is from the first axis. The point at which the axes cross is called the *origin*.

Here is a pair of axes, drawn in standard position. They are labeled. Whenever you make a graph on a plane using integers, you should first draw a set of axes and label them like this.



EXERCISES

1. Use graph paper. Draw neatly a pair of axes in standard position. Label them. Mark and label a dot for each of these ordered pairs of integers.

a. (1, 1)
b. (2, 1)
c. (3, 4)
d. (4, 6)
e. (1, 6)
f. (5, 2)
g. (0, 0)
h. ($^{-}3$, 5)
i. ($^{-}4$, 2)
j. ($^{-}5$, 5)
k. ($^{-}3$, 6)
l. ($^{-}1$, 7)
m. (4, $^{-}3$)
n. (6, $^{-}2$)
o. (3, $^{-}6$)
p. (4, $^{-}4$)
q. ($^{-}5$, $^{-}3$)
r. ($^{-}2$, $^{-}6$)
s. ($^{-}1$, $^{-}5$)
t. ($^{-}6$, $^{-}2$)
u. ($^{-}3$, $^{-}3$)
v. ($^{-}7$, $^{-}7$)
w. (0, 4)
x. ($^{-}5$, 0)
y. (0, $^{-}6$)
z. (3, 0)

2. What are the coordinates of points with these locations?
   a. Four units to the right and 5 units up.
   b. Five units to the left and 2 units up.
   c. Eight units to the left and 5 units down.
   d. Three units to the right and 7 units down.

3. a. If a point is on the first axis, what do you know about its second coordinate?
   b. If a point is on the second axis, what do you know about its first coordinate?
   c. If a point is on both axes, what do you know about its coordinates?

4. a. In what region of the plane do points have both coordinates positive?
   b. In what region do points have both coordinates negative?
   c. In what region do points have first coordinate positive?
   d. In what region do points have second coordinate negative?
   e. In what region do points have first coordinate negative and second coordinate positive?
   f. In what region do points have first coordinate positive and second coordinate negative?

5. What are the coordinates of the origin?

GRAPHS OF NUMBER SENTENCES

We have plotted graphs of number sentences before. Now let's use graphs on a plane. We shall study number sentences for integers. As before, the points of the graph will show the solution set of the sentence.

Let's study the number sentence for integers $p = q + 1$. Can we find a solution? How about the integer $^-3$? Where does it belong? Do we replace 'p' by '$^-3$' or replace 'q' by '$^-3$'? We see that we need a pair of integers—not just one. Perhaps we need an ordered pair of integers. Let's try $(^-3, 1)$. Now we have an integer for each variable of the sentence. But which one is which? We will have to decide. Since p comes before q in the alphabet, let's let it be first. This means 'p' is for the first coordinate. We make this agreement so we will all do it the same way.

Is $(^-3, 1)$ a solution of the sentence $p = q + 1$? If we replace 'p' by '$^-3$' and replace 'q' by '1', we get

$$^-3 = 1 + 1.$$

This equation is false. Therefore the ordered pair $(^-3, 1)$ is not a solution.

1.20 Let's Explore

Let's make a graph of the equation $p = q + 1$.

1. The ordered pair of integers (1, 0) is a solution. Check it.
2. Find six more ordered pairs in the solution set. Keep your results in a table, like this.

Example:

Replace 'p' by '2':

$$2 = q + 1$$

Now we see that to make a true sentence we replace 'q' by '1'. In the table write '2' for 'p' and under it write '1' for 'q'.

p	1	2	$^{-}1$	$^{-}2$	0	3	4
q	0	1					

3. Use graph paper. Draw a set of axes in standard position. Label the first axis "First (p)." Label the second axis "Second (q)."
4. Plot the points whose ordered pairs are given by your table.
5. Describe the graph. Plot some more points if you need to.
6. How many ordered pairs are in the solution set?

You have seen how to plot graphs of number sentences on a plane. We must decide which variable goes with the first axis, and which with the second axis. Then we find some ordered pairs in the solution set. We keep the results in a table. Then we plot the points. We also find some more if we need them.

The graph is a kind of picture of the solution set. As you plot graphs, look for patterns. Later you will learn more about patterns in graphs.

EXERCISES

Use graph paper. Draw axes in standard position. Plot graphs of these number sentences for integers.

1. $p = q$
2. $y = {}^{-}x$
3. $a < b$
4. $r \geq s$
5. $y = x - 3$
6. $y < x + 3$

*7. $t = 3$

8. $p = |q|$

*Hint: Use t for the first axis. Any ordered pair $(3, n)$ makes $t = 3$ true. The second coordinate does not matter.

DO YOU KNOW THESE TERMS?

To show that you know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

additive inverse (3)
negative integer (8)
positive integer (8)
inequality (32)
number phrase (35)
equivalent phrases (38)
simplify (40)
antecedent (41)
axis (49)
coordinates (49)
reciprocal (2)
integer (7)
absolute value (10)
negation (32)
expression (35)
conditional sentence (40)
consequent (41)
converse (45)
ordered pair (47)
origin (49)

CHAPTER SUMMARY

Mathematical system (1)
Integers (6)
Number line for integers (7)
Positive and negative integers (8)
Absolute value (10)
Addition for integers (11)
Subtraction (23)
Addition and subtraction on the number line (25)
Thermometers (28)
Other uses of the system of integers (29)
Number sentences (31)
Inequalities (32)
Negations (32)
Number phrases (34)
Equivalent phrases (37)
Simplifying phrases to solve equations (40)
Conditional sentences (40)
Converses (43)
Graphs on a plane (45)

REVIEW EXERCISES

1. Describe the set of integers.

2. In each of the following, place one of these symbols ('<', '>' or '=') to make a true statement:

   a. ${}^{+}6 \qquad {}^{+}3$
   b. ${}^{-}2 \qquad {}^{-}6$
   c. ${}^{-}3 \qquad {}^{+}3$
   d. ${}^{+}10 \qquad {}^{-}10$
   e. ${}^{+}10 + {}^{-}10 \qquad {}^{+}3 + {}^{-}3$
   f. ${}^{+}3 + {}^{-}3 \qquad {}^{+}3 - {}^{+}3$
   g. ${}^{-}2 + ({}^{+}3 + {}^{-}2) \qquad {}^{+}2 + ({}^{-}3 + {}^{+}2)$
   h. ${}^{-}({}^{-}17) \qquad {}^{-}7$
   i. ${}^{-}({}^{+}6) \qquad {}^{-}6$
   j. ${}^{+}11 \qquad {}^{-}({}^{-}11)$

3. What integer is neither positive nor negative?

4. Name the additive inverse of each of the following:

   a. ${}^{+}3$
   b. ${}^{-}2$
   c. 5
   d. 0
   e. ${}^{-}6$

5. Write the simplest name for each of the following integers:

   a. $3 + 5$
   b. ${}^{-}2 + {}^{+}7$
   c. ${}^{-}2 + {}^{-}5$
   d. ${}^{-}({}^{+}2)$
   e. ${}^{-}({}^{-}8)$
   f. ${}^{-}9 + {}^{+}9$
   g. ${}^{-}({}^{-}({}^{-}2))$
   h. ${}^{-}({}^{-}3 + 2)$
   i. ${}^{-}24 + {}^{+}6$
   j. ${}^{+}24 + {}^{-}0$

6. Which of the following are true:

   a. $|3| = |{}^{-}3|$
   b. $|{}^{-}8| = {}^{-}8$
   c. $|{}^{+}6| > |{}^{-}6|$
   d. $|{}^{-}2| < |{}^{+}8|$
   e. $|{}^{+}2| < |{}^{-}8|$

7. Find the solution set in the system of integers for each of the following:

   a. $x + 3 = 2$
   b. $x - 4 = 5$
   c. $b + {}^{-}6 = {}^{-}3$
   d. $t - {}^{-}2 = 0$
   e. ${}^{-}3 < x < {}^{+}6$
   f. $10 + c = {}^{-}20$
   g. ${}^{-}4 = a - 6$
   h. $2 + {}^{-}7 = d$
   i. $|f| < 2$
   j. ${}^{-}2 = {}^{-}3 + |y|$

8. Negate each of these sentences:
   - a. $3 + {}^-2 \neq 2$
   - b. ${}^-6 + {}^-9 = {}^-15$
   - c. $|4| = |{}^-4|$
   - d. $7 \leq {}^-9 + 16$
   - e. ${}^-2 \geq 5 + {}^-9$

9. Which of the following are number phrases?
   - a. 2
   - b. $x + 3$
   - c. $\frac{5}{a}$
   - d. $\frac{9}{x - x}$
   - e. $\frac{a - a}{2}$

10. Simplify each of the following phrases:
    - a. $(y - 2) + {}^-y$
    - b. ${}^-3 + {}^+2 + b$
    - c. ${}^-(x + 2) + x - 2 + x$
    - d. $(y - 4) + 4$
    - e. ${}^-(7 - x) - {}^-7$

11. Make up 5 conditional sentences. Underline the antecedent with one line, the consequent with two lines.

12. When is a conditional sentence true?

13. Write the converse of each of the following:
    - a. $3 < x \rightarrow x \geq 4$
    - b. $x = 2 \rightarrow x + 2 = 4$

14. Plot the points having the following coordinates, on one pair of axes.
    - a. $(3, 4)$
    - b. $({}^-2, 5)$
    - c. $({}^-1, {}^-6)$
    - d. $(2, 2)$
    - e. $({}^-5, {}^-2)$

15. Plot the graph of:
    - a. $x = 2$
    - b. $y = {}^-1$
    - c. $x = {}^-y$

16. Draw a line segment.

17. Draw a line segment open at one end.

18. a. Draw two rays with a common end point.
    b. What is another name for this figure?

19. Draw examples of each of the following:
    - a. closed curve
    - b. a non-simple quadrilateral
    - c. a convex hexagon
    - d. a regular triangle
    - e. a concave pentagon
    - f. a right triangle
    - g. an obtuse angle
    - h. an isosceles triangle
    - i. supplementary angles
    - j. a polygon which has no diagonals

20. Write the prime factorization of each of the following numbers:

    a. 175 b. 96 c. 242 d. 60 e. 735

21. Find the greatest common factor of 75, 125 and 1000.

22. Find the greatest common factor of 49, 25 and 96.

23. What number is the least common multiple of 2, 4 and 8?

24. What number is the least common multiple of 15, 21, and 24?

25. Which of the following numbers is divisible by 2? By 3. By 4. By 5. By 6. By 9. By 10. By 12.

    a. 3464 b. 24,895 c. 23,400 d. 17,001 e. 392

26. Describe the following sets:
    - a. $A \cap B$ if $A = \{2, 4, 6, \cdots\}$ and $B = \{1, 2, 3, \cdots\}$.
    - b. The union of the set of odd integers and the set of prime natural numbers.
    - c. In the set of natural numbers, the intersection of the set of divisors of 50 and the set of multiples of 2.
    - d. The intersection of the set of reciprocals of natural numbers and the set of integers.
    - e. The intersection of the set of regular polygons and the set of quadrilaterals.

27. Write the simplest fractional numeral for each of the following arithmetic numbers:
    - a. $\frac{3}{4} \times \frac{3}{5}$
    - b. $\dfrac{2 + 1\frac{3}{4}}{3\frac{1}{4} - 1\frac{1}{2}}$
    - c. 5% of 2
    - d. $.04 + \frac{2}{5}$
    - e. 25% of 2

READING LIST

Dubisch, R. *The Nature of Number*. New York: The Ronald Press Company, 1952. pp. 50-59.

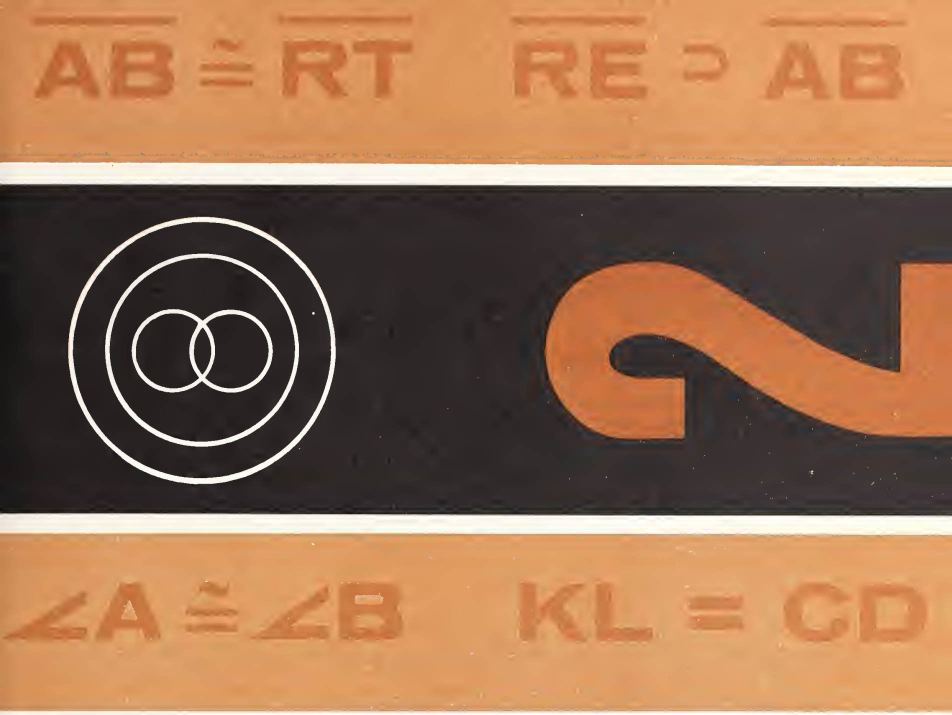
University of Maryland Mathematics Project. *Mathematics for the Junior High School, First Book*. College Park: Univ. of Md., 1959. pp. 195-216.

$\overline{AB} \cup \overline{CD} = \overline{CD}$

2

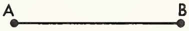
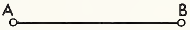
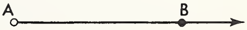
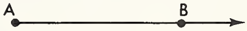
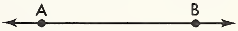
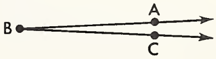
CHAPTER TWO

Geometry: Congruence, Constructions and Circles


$\overline{AB} \cong \overline{RT}$
$\overline{RE} \supset \overline{AB}$
2
$\angle A \cong \angle B$
$KL = CD$

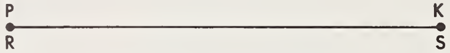
GEOMETRIC FIGURES

In geometry we study *figures*. Remember that a figure is a *set of points*. Let's review some of the special kinds of geometric figures we have studied. The following list gives the kind of figure, and also shows how each kind is named and how each kind is drawn.

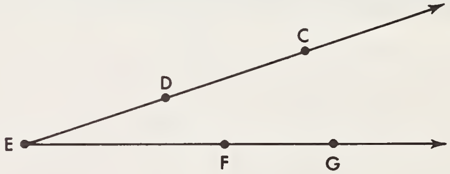
Kind of Figure	How Named	How Drawn
1. Segment	$\overline{AB}$	
2. Open segment	$\overset{\circ\!-\!\circ}{AB}$	
3. Half-line	$\overset{\circ\!\longrightarrow}{AB}$	
4. Ray	$\overrightarrow{AB}$	
5. Line	$\overleftrightarrow{AB}$	
6. Angle	$\angle ABC$ or $\angle B$	

What is the definition of *segment*? *ray*? *open segment*? *half-line*? *angle*?

A sentence with '=' for its verb says that the symbols on either side of the verb name the same things. The sentence $\overline{PK} = \overline{RS}$ says that '$\overline{PK}$' and '$\overline{RS}$' are two names for the same segment. If the sentence refers to this figure it is true:



The following angle can be named several ways.



These sentences about this angle are true:

$$\angle CEG = \angle DEF$$
$$\angle DEG = \angle CEF$$
$$\angle CEF = \angle E$$

Some sets of points, or figures, have measures. Measures are numbers. If we want to say that two sets of points have the same measure we may write sentences such as the following:

$$m\ \overline{KL} = m\ \overline{CD}$$
$$KL = CD$$

These sentences are about the measures of segments $\overline{KL}$ and $\overline{CD}$. They say that the two segments have the same number for a measure, or *length*. The sentence $m\ \angle ABC = 30°$ says that the measure of angle ABC is 30°.

Since in geometry we study sets of points, we often use set notation. Let's review the meaning of some of the symbols used with sets. Suppose

$$A = \{5, 10, 15, 20\}, \text{ and}$$
$$B = \{5, 8, 10, 15, 18, 20\}.$$

Every member of A is also a member of B. We say that A is *included* in B, or is a *subset* of B. We may write the following sentence to say this:

$$A \subset B$$

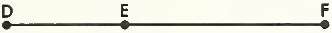
You should recall that every set is a subset of itself. In other words,

If $A = B$, then $A \subset B$ (and also $B \subset A$)

You should also recall that the empty set (which has no members) is a subset of every set. In other words,

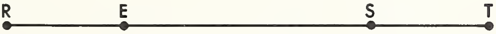
For any set A, $\phi \subset A$.

Note: This means that every member of ϕ is also a member of A. (Of course ϕ has no members.) In this figure, every point of $\overline{DE}$ is also a point of $\overline{DF}$.



Then $\overline{DE} \subset \overline{DF}$.

You have also worked with intersections of sets and unions of sets. For the intersection of sets A and B we write $A \cap B$'. For the union of sets A and B we write '$A \cup B$'. In this figure, $\overline{RS} \cap \overline{ET} = \overline{ES}$.



Also, $\overline{RE} \cup \overline{RS} = \overline{RS}$.

Now that you have reviewed some of the basic ideas of figures on a plane, let's learn more about them.

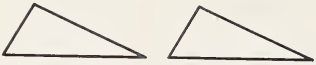
CONGRUENCE

One reason geometry is important and useful is that it gives us a way of telling when and in what ways sets of points are alike. You already know, for example, that certain polygons are alike because they have three sides. We call polygons like these *triangles*. Some pairs of intersecting lines are alike because they form right angles. We call such lines *perpendicular* lines.

2.1 Let's Explore

1. At the top of the next page are some pairs of geometric figures. In each case, tell whether or not one figure will fit exactly on the other one if you slide one over the other without rotation.

a.



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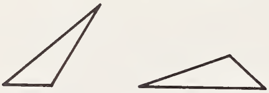

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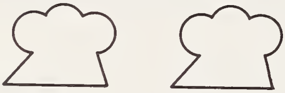

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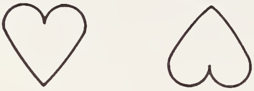

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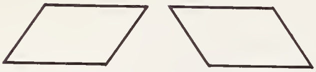

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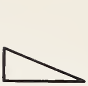

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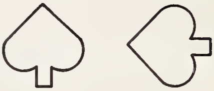

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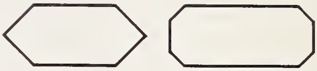

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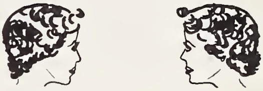

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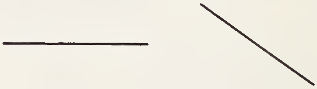
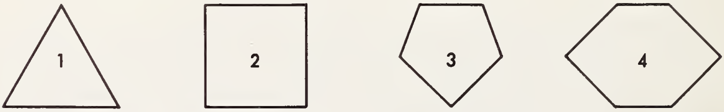


2. In each case of exercise 1, which pairs of figures would fit exactly if you could rotate one figure, as well as slide it? (You may trace over figures on a thin piece of paper and then turn and slide the paper if you need to.)

3. In each case of exercise 1, which pairs of figures would fit exactly if you could flip them as well as turn and slide them? (You may trace over figures and cut them out, and then turn and slide and flip them, if you need to.)

4. In each case, pick out the figures that would fit each other exactly if you could move them in any way. (You may trace them and cut them out if you need to.)

a.



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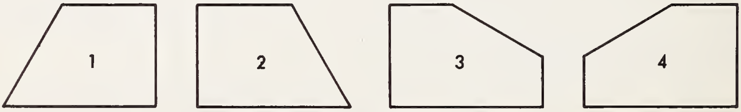

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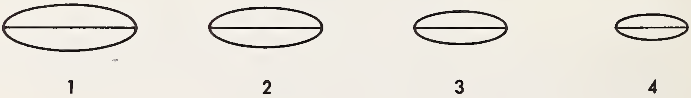

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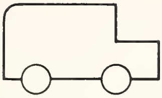
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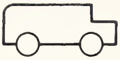
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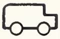
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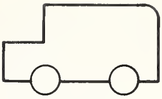
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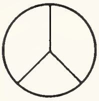
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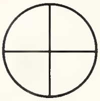
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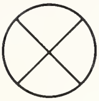
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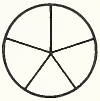
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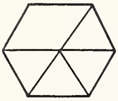
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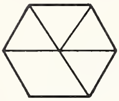
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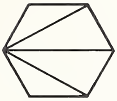
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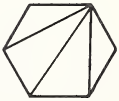
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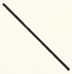
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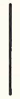
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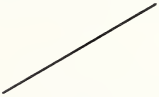
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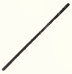
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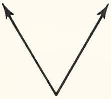
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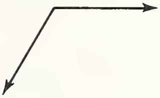
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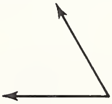
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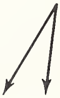
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5. Draw a segment 2 inches long. Draw another segment that would fit on it exactly.

6. When will two segments fit exactly if they are placed together?

7. Draw an angle, with your protractor, whose measure is 35°. Draw another angle that would fit on it exactly.

8. When will two angles fit exactly if they are placed together?

Geometric figures that would fit exactly if placed together are called *congruent* figures. When two figures are congruent they have the same size and shape.

Since all segments have the same shape, two segments will be congruent if and only if they have the same measure. If a segment $\overline{AB}$ has a measure of 11 inches and segment $\overline{RT}$ has a measure of 11 inches, $\overline{AB}$ will fit exactly on $\overline{RT}$. Then $\overline{AB}$ is congruent to $\overline{RT}$. We have a new mathematical verb, "is congruent to." We use the symbol '≅' for it. To say that $\overline{AB}$ is congruent to $\overline{RT}$ we write the sentence

$$\overline{AB} \cong \overline{RT}.$$

All angles consist of two rays. Some pairs of angles are congruent and some are not. If two angles each have a measure of 30° then they will fit together exactly. The sentence

$$\angle A \cong \angle B$$

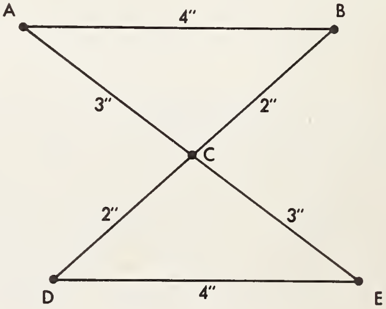
says that the angles A and B are congruent. They will fit together exactly.

Two geometric figures are ***congruent*** *if and only if they will fit together exactly.*
If two segments have the same measure then they are congruent.
If two angles have the same measure then they are congruent.

You should remember that congruent figures have the same measure. The converse is not true. Some pairs of figures may have the same measure but not be congruent. You will learn more about this later.

EXERCISES

1. Draw a segment 3 inches long. Draw another segment congruent to it.
2. Draw an angle whose measure is 100°. Draw another angle congruent to it.
3. Draw a triangle. Draw another triangle congruent to it.
4. Draw a quadrilateral. Draw another quadrilateral congruent to it.
5. Which of these sentences about the accompanying figure are true?
   a. $\overline{AB} = \overline{DE}$
   b. $\overline{AC} \cong \overline{CE}$
   c. $DC \cong BC$
   d. $m\,\overline{AB} \cong m\,\overline{DE}$
   e. $\angle ACB = \angle DCE$
   f. $\angle ACB \cong \angle DCE$
   g. $\angle ACB = \angle BCA$
   h. $\angle DCE \cong \angle ECD$
   i. $\triangle ACB \cong \triangle ECD$

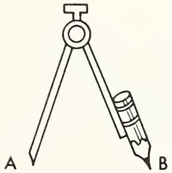


COMPASS AND STRAIGHTEDGE

In your study of geometry, you have used two instruments; the protractor, to measure angles, and the ruler, to measure distances.

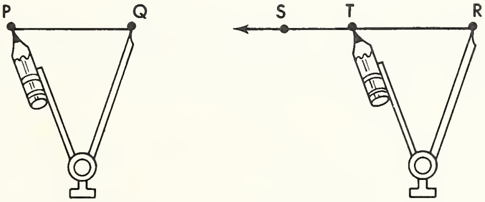
Students of geometry in ancient Greece used two different instruments. They used the *straightedge* and *compass.*

A straightedge is used to draw segments. A ruler is a straightedge with a scale. Whenever you use a ruler to draw a segment, you are using it as a straightedge.



A compass is used to mark off distances. It looks like this. It has two arms which rotate around the top, where they are joined. On the end of one arm is a metal tip, or point (A). On the end of the other arm is a pencil or pen (B). The distance marked off is the distance between the two points A and B.

Suppose you want to draw a segment on $\overrightarrow{RS}$ which is congruent to a certain segment $\overline{PQ}$.

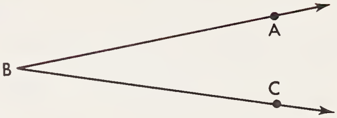


First you place the compass so that one point is on P while the other point is on Q, as in the picture. Then you pick up the compass, being careful not to change the distance between the ends, and place the metal tip on R. Then swing the compass in position so that the pencil marks a point T on $\overrightarrow{RS}$. Sometimes it is easier to draw a larger mark. T will be the intersection of the mark with $\overrightarrow{RS}$, as in the picture. Then $\overline{RT}$ is congruent to $\overline{PQ}$. ($\overline{RT} \cong \overline{PQ}$).

In your study of geometry, you will use the compass, as well as the straightedge and protractor. You will learn how to use these instruments to draw fairly accurate geometric figures. Whenever we do this, we say we are making a *construction.* Later on when you study formal geometry you will prove that these constructions are correct.

CONSTRUCTION: TO COPY AN ANGLE

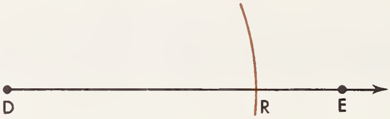
Here is an angle. How could you



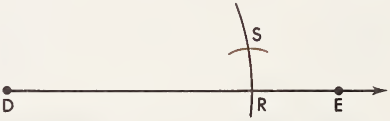
construct another angle congruent to it? This would be easy if you had a protractor. Can you do it with a compass and straightedge?

2.2 Let's Explore

1. Draw a ray $\overrightarrow{DE}$. Now open your compass to about $1\frac{1}{2}$ inches and make a mark, as shown.

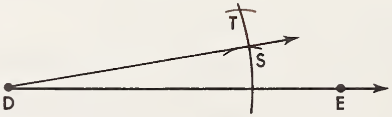


2. Set your compass at about a $\frac{1}{4}$ inch distance. Put the metal tip at R on your drawing. Make a mark which meets the other one, as shown. Call the point where the marks meet 'S'.



Now draw ray $\overrightarrow{DS}$. Do you have an angle? Name it.

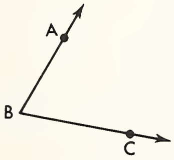
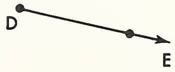
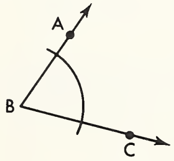
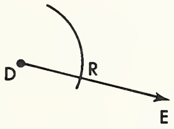
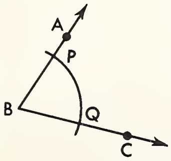
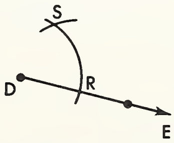
3. Set your compass at about a $\frac{1}{2}$ inch distance. Put the metal tip at R again, and make a mark, as before. Call the point where the marks meet 'T'.

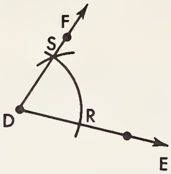


Draw ray $\overrightarrow{DT}$.

4. Set your compass at about a 1 inch distance. Put the metal tip at *R* again and make another mark. Call the intersection of the marks '*U*'. Draw ray $\overrightarrow{DU}$.
5. Set your compass at about a 2 inch distance. Put the metal tip at *R* again and make another mark. Call the intersection of the marks '*V*'. Draw ray $\overrightarrow{DV}$.
6. Compare the sizes of the angles *EDS*, *EDT*, *EDU* and *EDV*. Which is largest? Which is smallest?
7. For each opening of your compass do you get a different size angle?

You have now seen that once we draw a curve with the metal tip at the endpoint of a ray, we can construct different size angles with different compass openings. This idea gives us a way to construct an angle congruent to any given angle. Study the following instructions to see how it can be done.

Old Angle	New Angle	Description
		1. Draw a ray $\overrightarrow{DE}$, where you want the second angle to be. Decide what point will be its vertex. See if you can picture the angle in your mind.
		2. Place the metal tip on B. Make a mark that meets the sides of the angle. Do not change the compass setting. Put the metal tip on the vertex, *D*, and make a similar mark meeting $\overrightarrow{DE}$.
		3. Place the compass on *P* and *Q* to get the distance. Do not change the compass setting. Place metal tip on *R* and make a mark *S*.

		4. Draw $\overrightarrow{DS}$. Then $\angle FDE \cong \angle ABC$.

Draw several different angles and copy them using a compass.

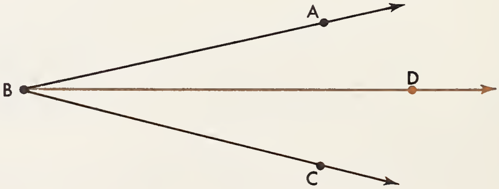
Let's Estimate

Remember that estimating is important. You should practice it often. Try these.

1. $486 + 214$
2. $3{,}467{,}105 + 317{,}702$
3. $1704 - 517$
4. $427{,}613 - 13{,}852$
5. $613 \cdot 11$
6. $339 \cdot 19$
7. $\frac{306}{97}$
8. $\dfrac{49.2}{2.1}$
9. $\dfrac{76.2}{26.1}$
10. $\dfrac{1.882}{9.302}$
11. $471 - 568$
12. $^-173 + 25$
13. $^-89 + {}^-488$
14. $1412 + {}^-492$

CONSTRUCTION: TO BISECT AN ANGLE

To *bisect* means to cut in half. An angle is bisected by drawing a ray between the sides of the angle so that two smaller angles with the same measure are formed. Here $\angle ABC$ is bisected by $\overrightarrow{BD}$, so $\angle ABD \cong \angle DBC$. The ray $\overrightarrow{BD}$ is called the *angle bisector*.



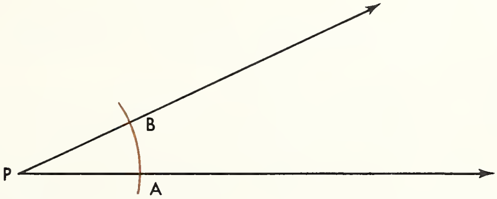
An angle can easily be bisected with a protractor. How? The ray $\overrightarrow{BD}$ must be drawn so that $m\ \angle ABD = \frac{1}{2} \cdot m\ \angle ABC$. If $m\ \angle ABC = 60°$ then $m\ \angle ABD = 30°$ and $m\ \angle DBC = 30°$.

An angle bisector can also be found using a compass and straightedge.

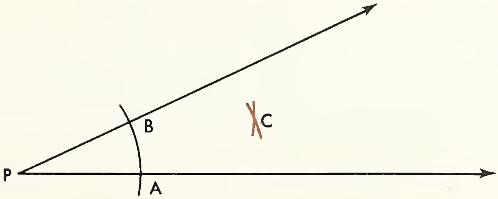
2.3 Let's Explore

Be sure you have a sharp pencil. Also be sure the pencil on your compass is sharp. Do the following very carefully and neatly.

1. Draw an angle whose measure is about 60° or 65°.
2. Carefully draw another angle congruent to it.
3. Bisect the first angle using your protractor.
4. Label the vertex of the other angle '*P*'. Open your compass to about $\frac{3}{4}$ inch. Make a mark as shown. Label the points '*A*' and '*B*' as shown.



5. Put the metal tip of your compass on *A* and the pencil on *B*. Then make a mark about in the middle of the angle.
6. Don't change the compass setting. Put the metal tip on *B* and make a mark that meets your other one, as shown. Call the point where the marks intersect '*C*'.



7. Open your compass about $\frac{1}{4}$ inch wider, and then repeat part 6. Call the new point of intersection '*D*'.

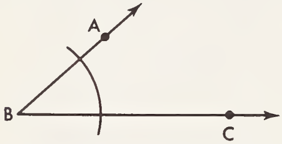
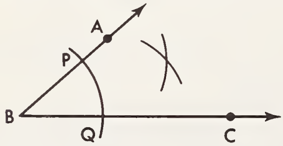
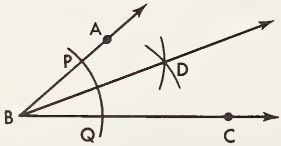
8. Open your compass about another $\frac{1}{4}$ inch, and repeat part 6. Call the new point of intersection 'E'.

9. Open your compass about another $\frac{1}{4}$ inch, and repeat part 6. Call the new point of intersection 'F'.

10. What do you notice about the points P, C, D, E and F?

If you carried out instructions carefully you probably discovered that the points P, C, D, E and F are all on a line. The ray $\overrightarrow{PC}$ contains the points D, E and F. You probably also suspected that the ray $\overrightarrow{PC}$ is the bisector of the angle. Measure the two small angles and see if this is true.

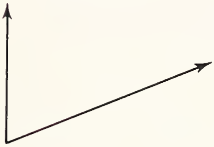
The exploring you just did shows a way to bisect any angle. Study the following series of diagrams. They show you how this can be done.

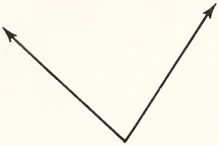
Angle	Description
	1. Place the metal tip of the compass on B and make a mark that meets the sides of the angle.
	2. Place the metal tip on P and make a mark about where you think the bisector should be. Place the metal tip on Q and do the same thing.
	These marks should intersect.
	3. Draw $\overrightarrow{BD}$. The two angles, $\angle ABD$ and $\angle DBC$ are congruent. They have the same measure. $\overrightarrow{BD}$ bisects the $\angle ABC$.

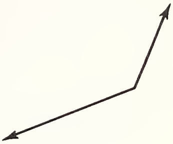
Draw several different angles. Bisect them with compass and straightedge.

EXERCISES

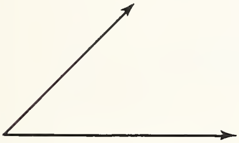
1. Using a protractor, draw angles having the same measure as these angles. Copy them, using compass and straightedge. Check your work with a protractor.

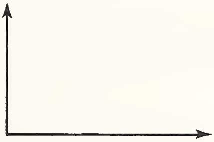


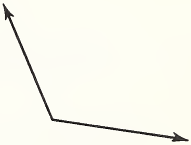




2. Using a protractor, draw angles having the same measure as these angles. Copy them, using compass and straightedge. Bisect each angle. Check your work with a protractor.



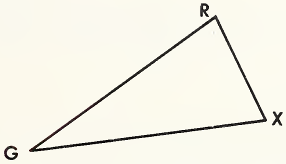




3. Draw an angle whose measure is 135°, using a protractor. Bisect it, using compass and straightedge.

4. You know how to copy segments and angles. Study this example to see how a triangle may be copied by copying some of its sides and angles.

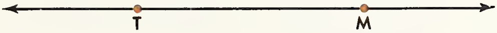
Example:



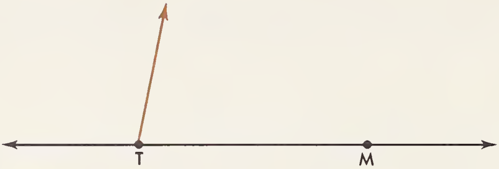
Triangle GRX can be copied by copying $\angle G$, $\angle R$ and $\overline{GR}$.

First draw a line on which to build the triangle.

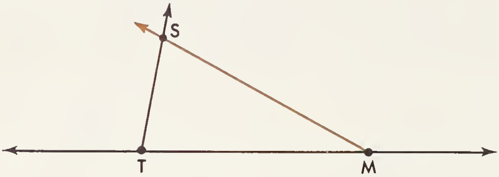
Now make a segment congruent to $\overline{GR}$ on that line.



Construct an angle congruent to $\angle R$, with vertex at T.

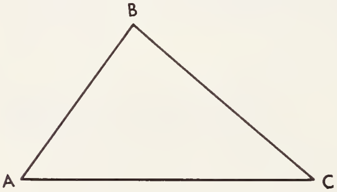


Construct an angle congruent to $\angle G$, with vertex at M.



Triangle MTS is congruent to triangle GRX.

Now look at this triangle.



a. Draw a triangle similar to this.

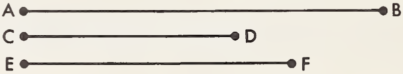
b. Copy that triangle by copying $\overline{AC}$, $\angle BAC$, and $\angle BCA$.

c. Copy your triangle by copying $\overline{AC}$, $\angle BAC$, and $\overline{AB}$.

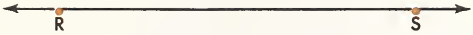
5. A triangle may be constructed so that its sides are congruent to three given segments. Study this example:

Example:

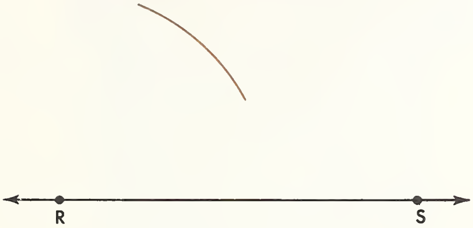
Construct a triangle whose sides are congruent to segments $\overline{AB}$, $\overline{CD}$ and $\overline{EF}$.



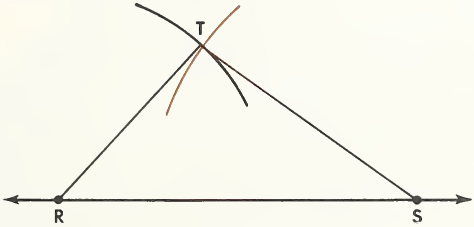
First draw a line on which to build the triangle. Make a segment congruent to $\overline{AB}$ on that line.



Open your compass to the distance CD. Place the metal tip on R and make a mark. The third vertex will be somewhere on this mark.

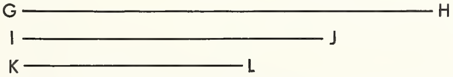


Open your compass to the distance EF. Place the metal tip on S and make a mark. It should intersect the first mark. This intersection is the third vertex.



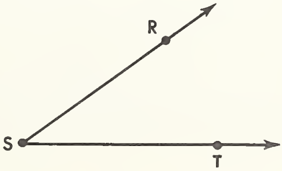
Then draw the sides $\overline{RT}$ and $\overline{ST}$.

Now construct a triangle whose sides are congruent to these segments.



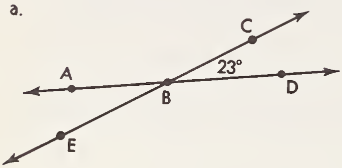
6. Draw a triangle. Now construct another triangle congruent to it by copying all three sides.

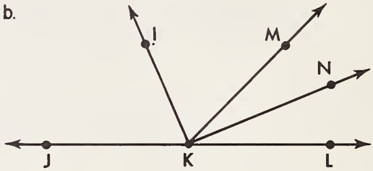
7. Construct an isosceles triangle (a triangle with two sides congruent) having the congruent sides 3″ long and the angle formed by these sides congruent to $\angle RST$.



8. Remember two angles are supplementary when the sum of their measures is 180°. Find the measure of a supplement of an angle whose measure is:
   a. 35° b. 106° c. 91° d. $\frac{1}{2}°$ e. $58\frac{1}{4}°$

9. Find the measures of as many angles as you can, without using a protractor.





$m\ \angle MKL = 46°$

$\overrightarrow{KI}$ bisects $\angle MKJ$

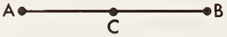
$\overrightarrow{KN}$ bisects $\angle MKL$

10. Draw a triangle. Bisect its angles, using a compass.

11. a. Draw a straight angle. Bisect it, using a compass.
    b. Describe the relationship between the bisector and the sides of the angle.

CONSTRUCTION: TO BISECT A SEGMENT

Here is a segment.



There is a point C which divides the segment into two congruent segments $\overline{AC}$ and $\overline{CB}$. The point C is called the *midpoint* of $\overline{AB}$. Any line which does not contain $\overline{AB}$ and which contains its midpoint is called a *bisector* of the segment. In figure 1 below $\overleftrightarrow{RS}$ is a bisector of $\overline{JK}$. Figure 2 below shows that a segment has many bisectors.

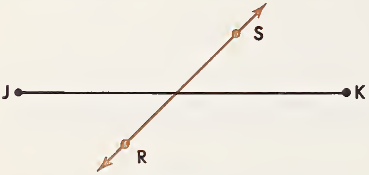


Figure 1

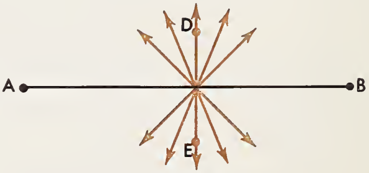


Figure 2

One of the bisectors of $\overline{AB}$ is perpendicular to $\overline{AB}$ because it forms a right angle with the segment. This line, $\overleftrightarrow{DE}$, is called the *perpendicular bisector* of AB.

Carl Friedrich Gauss

1777 - 1855

Gauss is probably the greatest mathematician that Germany ever produced. He was an astronomer and a physicist as well. He was one of the first to apply mathematical ideas of probability to astronomy in a way that foresaw modern statistics. He invented the mathematical system of complex numbers, which you will study later.

Gauss also proved that it is impossible to trisect an angle using a straightedge and compass alone. He did this by showing that all constructions with straightedge and compass could be described by number sentences involving at most an exponent of 2, such as $x^2 + 2x - 3 = 0$, but the trisection of an angle is described by an equation requiring an exponent of 3, such as $x^3 + 5x = 0$. No straightedge and compass construction would give this. Therefore it is impossible.

We can construct the perpendicular bisector of any segment using a compass and straightedge.

2.4 Let's Explore

Be sure you have a sharp pencil. Also be sure the pencil on your compass is sharp. Do the following very carefully and neatly.

1. Using your ruler, draw a segment two inches long. Mark its midpoint 'T'.
2. Using your protractor, draw a line $\overleftrightarrow{PQ}$ on T and perpendicular to the segment. You have now drawn the *perpendicular* bisector of the segment.
3. Draw another segment two inches long. Label its endpoints 'A' and 'B'.
4. Open your compass to about $1\frac{1}{2}$ inches. Place the metal tip at A and make two marks, as shown in figure 1 below.

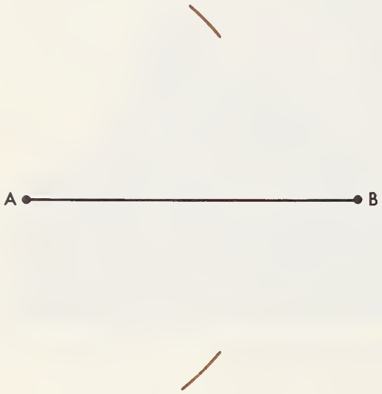


Figure 1

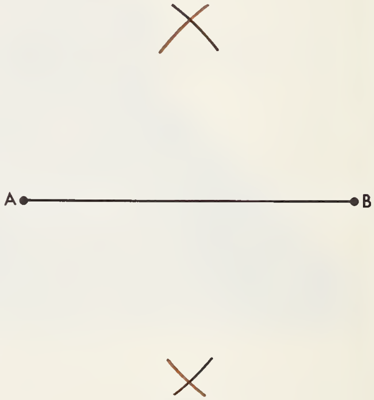


Figure 2

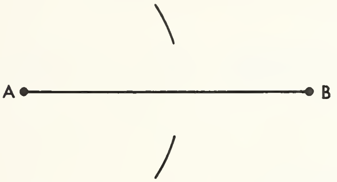
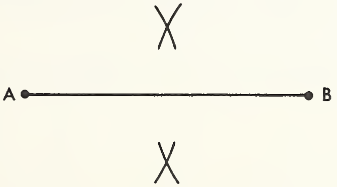
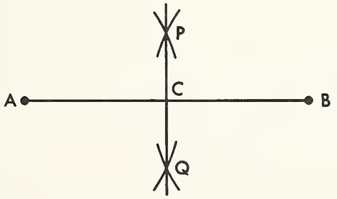
5. Do not change the compass setting. Place the metal tip at B and make two more marks, intersecting the first ones as shown in figure 2 above.
6. Open your compass to about 2 inches. Repeat parts 4 and 5.
7. Open your compass to about $2\frac{1}{2}$ inches. Repeat parts 4 and 5.
8. What do you notice about the intersections of the marks you have made?
9. Try to draw a straight line which contains all of the intersections. Find the point where it meets $\overline{AB}$. Label that point 'T'.
10. Use your compass. Find out whether $\overline{AT} \cong \overline{TB}$.

11. Is your line a bisector of $\overline{AB}$?

12. Measure one of the angles formed by the line and the segment.

13. Is your line the perpendicular bisector of $\overline{AB}$?

If you carried out instructions carefully you probably discovered how to construct the perpendicular bisector of a segment. To make sure you understand, study the following series of diagrams.

Construction	Description
	1. The compass setting must be greater than $\frac{1}{2}$ AB. With this setting, place the metal tip at A and make marks above and below the segment.
	2. Do not change the compass setting. Place the metal tip on B, and do the same thing. These marks should intersect the other marks.
	3. Draw $\overleftrightarrow{PQ}$. It will intersect $\overline{AB}$ at its midpoint. $\overleftrightarrow{PQ}$ bisects $\overline{AB}$. It is also perpendicular to $\overline{AB}$. $\overline{PQ}$ is the *perpendicular bisector* of $\overline{AB}$.

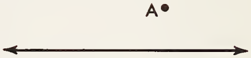
Draw several different segments, and bisect them using compass and straightedge.

CONSTRUCTION: TO DRAW A LINE PERPENDICULAR TO A LINE AT A POINT ON THE LINE

Here is a line and a point on the line. It is possible to draw a line, a ray, or a segment perpendicular to the line at the point A. Whenever we make this construction, we say we are *drawing a perpendicular*. Consider the line as a straight angle whose vertex is A. The angle bisector then is perpendicular to the line at A. Why? To construct the perpendicular, you construct the angle bisector of the straight angle whose vertex is A.

Draw several lines with points on them, and construct the perpendiculars to the lines at the given points.

CONSTRUCTION: TO DRAW A PERPENDICULAR TO A LINE FROM A POINT NOT ON THE LINE



Here is a line and a point not on the line. It is possible to construct a line on A which is perpendicular to the line.

2.5 Let's Explore

Be sure you have a sharp pencil. Also be sure the pencil on your compass is sharp. Do the following very carefully and neatly.

1. Draw a line and a point A about 1 inch from the line, as in figure 1 below.

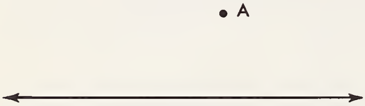


Figure 1

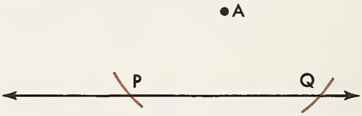
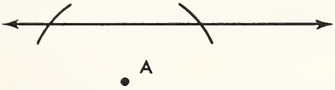
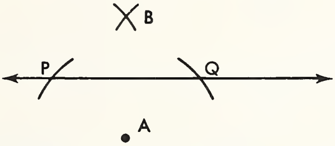
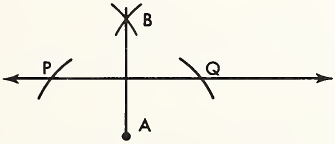


Figure 2

2. Open your compass to about 2 inches. Make two marks which intersect the line, as in figure 2 above. Call them 'P' and 'Q'.
3. Construct the perpendicular bisector of segment $\overline{PQ}$.
4. Does the perpendicular bisector contain the point A?
5. If your perpendicular bisector does not contain the point A, your drawing was not done carefully enough. Do it over, more carefully.

The exploring you have just done should give you an idea about this construction. It can be done just a bit more simply. Study the following diagrams to see how this construction should be done.

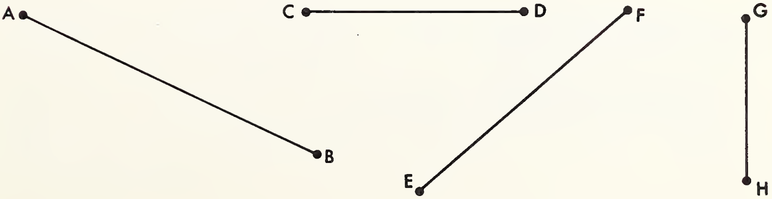
Construction	Description
	1. Place the metal tip of the compass on A and make two marks which intersect the line.
	2. Place the metal tip first on P and then on Q, and without changing the compass setting, make intersecting marks at B.
	3. Draw $\overleftrightarrow{AB}$. It will be perpendicular to the line.

Draw several lines with points not on the line and construct perpendiculars to the lines from these points.

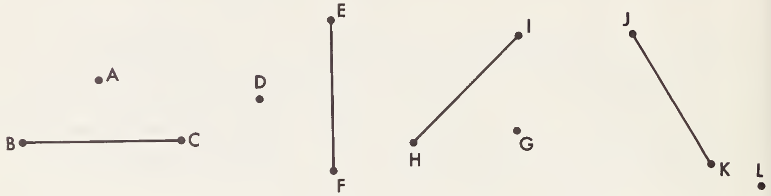
EXERCISES

In these exercises, do not use a protractor.

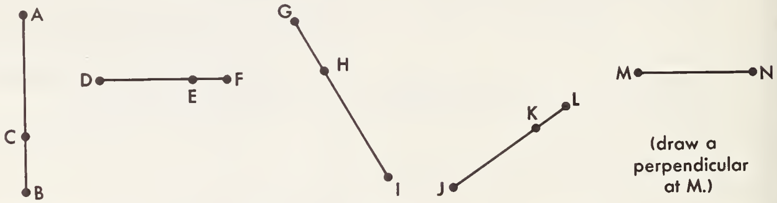
1. Here are some segments. Copy them using a compass. Then bisect them.



2. Copy these segments, and construct perpendiculars to them at the indicated points.



3. Copy these segments and points, and construct perpendiculars to the segments from the points.



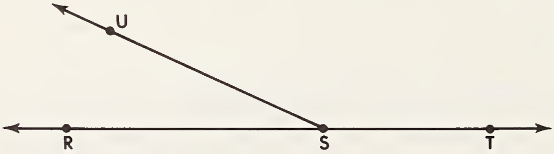
(draw a perpendicular at M.)

4. Draw a segment. Divide it into four congruent segments.

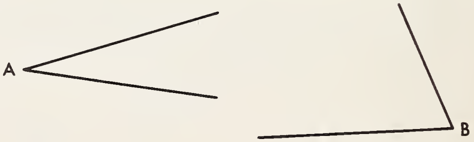
5. Draw an obtuse angle (one whose measure is more than 90°). Divide it into four congruent angles.

6. Draw a right triangle having the sides on the right angle 2″ and 3″ long.

7. Copy the picture below. Bisect $\angle RSU$. Bisect $\angle UST$. Describe the relationship between the bisectors. Can you prove your answer?



8. Construct $\triangle ABC$ so that $AB = 4$ inches and $\angle A$ and $\angle B$ are congruent to these angles.



9. Draw a segment. Construct its perpendicular bisector. Are there other lines which bisect the segment? Draw some of them.

10. Draw a triangle. Construct the perpendicular bisectors of the three sides of the triangle.

Review Practice

Do these calculations, to keep in practice. Remember always to estimate an answer before you begin to calculate with pencil and paper.

1. Add:

a. $\begin{array}{r}356\\210\\490\\\underline{243}\end{array}$ b. $\begin{array}{r}9842\\5111\\2983\\\underline{4005}\end{array}$ c. $\begin{array}{r}514\\603\\^{-}102\\\underline{^{-}498}\end{array}$ d. $\begin{array}{r}6512\\^{-}7493\\3014\\\underline{^{-}9812}\end{array}$

2. Multiply:

a. $\begin{array}{r}482\\\underline{35}\end{array}$ b. $\begin{array}{r}9422\\\underline{68}\end{array}$ c. $\begin{array}{r}4.312\\\underline{21.3}\end{array}$ d. $\begin{array}{r}17.815\\\underline{43.61}\end{array}$

3. Add or subtract, as shown:

a. $\frac{3}{5}+\frac{1}{5}+\frac{2}{5}$ d. $\frac{5}{7}-\frac{3}{21}$

b. $\frac{3}{8}+\frac{1}{4}+\frac{1}{2}$ e. $\frac{4}{9}+\frac{5}{7}-\frac{1}{3}$

c. $\frac{2}{3}+\frac{1}{6}+\frac{5}{12}$ f. $1\frac{2}{3}+4\frac{3}{4}-2\frac{1}{2}$

4. Multiply or divide, as shown, and simplify:

a. $\frac{2}{3}\cdot\frac{4}{5}$ d. $\frac{5}{9}\div\frac{3}{5}$

b. $\frac{7}{5}\cdot\frac{3}{10}$ e. $3\frac{1}{2}\cdot 1\frac{1}{3}$

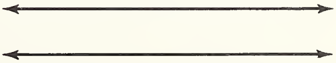
c. $\dfrac{\frac{8}{25}}{\frac{5}{2}}$ f. $1\frac{2}{3}\div 1\frac{1}{4}$

5. Divide:

a. $27\overline{)452}$ c. $1.34\overline{)16.08}$

b. $3.5\overline{)70.7}$ d. $17.6\overline{)6.16}$

PARALLEL LINES

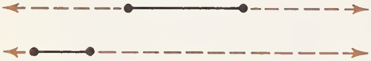


Here are two lines on a plane. They do not intersect. Their intersection is the empty set. Such lines are called *parallel* lines.

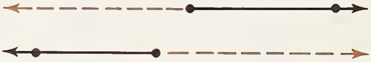
***Parallel lines** are lines which*
1. are on the same plane, and
2. do not intersect.

Can you find a pair of lines which do not intersect and yet are not parallel? Such lines are called *skew lines*.

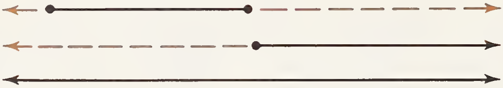
Rays, segments, and lines are parallel if they are on parallel lines. Here are a few examples of parallel sets.



two parallel segments



two parallel rays



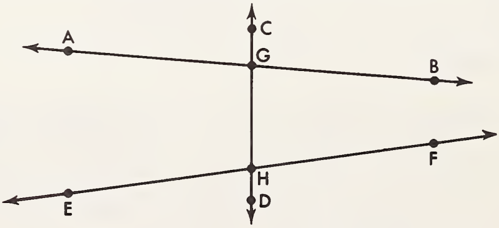
a segment, a ray, and a line which are parallel

To say that $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are parallel we can write the sentence $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$. The symbol '||' means *is parallel to*. If $\overleftrightarrow{AB} \parallel \overleftrightarrow{CD}$, is $\overleftrightarrow{CD} \parallel \overleftrightarrow{AB}$?

You will want to discover some relationships that exist when lines are parallel. To do this, you will need to learn some more words.

TRANSVERSALS

If a line intersects two lines, we call that line a *transversal*. In this picture, $\overleftrightarrow{CD}$ intersects $\overleftrightarrow{AB}$ and $\overleftrightarrow{EF}$. $\overleftrightarrow{CD}$ is a transversal. The transversal and the two lines form several angles. We shall now study them.



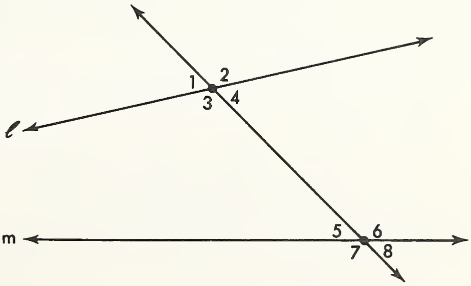
2.6 Let's Explore

1. Draw two lines and a transversal, as in the picture above. Label points of your drawing in the same way.
2. List all the angles shown on your drawing having G as a vertex. (Do not list any straight angles.)
3. List all the angles shown on your drawing having H as a vertex. (Do not list any straight angles.)
4. Which two angles at G have $\overrightarrow{GB}$ for one side?
   We say that these angles are *on the same side* of the transversal.
5. Which two angles at H have $\overrightarrow{HF}$ for one side?
   We say that these angles are *on the same side* of the transversal. They are also on the same side as angles CGB and HGB.
6. Now let's look on the other side of the transversal.
   a. Which two angles at G are on that side of the transversal?
   b. Which two angles at H are on that side of the transversal?
7. Look at angles DHF and AGC. Are they on the same side of the transversal or on opposite sides?
8. Look at angles HGB and GHF. Are they on the same side of the transversal or on opposite sides?

You have now seen how we speak of angles as being on the same side of a transversal or on opposite sides of a transversal. If the angles open toward the same side of the transversal then we say they are "on the same side" of the transversal.

2.7 Let's Explore

1. Here are two lines and a transversal. This time we shall name the angles by writing a numeral between the rays of the angle. Copy this drawing on your paper.



2. Find all the angles which open between the lines l and m. These angles are called *interior* angles.
3. The other angles are called *exterior* angles. Name all the exterior angles.
4. Find a pair of interior angles on the same side of the transversal.
5. Find a pair of exterior angles on opposite sides of the transversal.
6. Find a pair of interior angles on opposite sides of the transversal that do not have the same vertex. Such angles are called *alternate* interior angles.
7. Find another pair of alternate interior angles.
8. Find a pair of exterior angles on opposite sides of the transversal that do not have the same vertex. Such angles are called *alternate* exterior angles.
9. Find another pair of alternate exterior angles.
10. On your drawing shade the inside of a pair of alternate interior angles.
11. Draw another picture like the one above. Label it the same way.
12. Find an angle which opens above line l. Shade the inside of this angle.
13. Find an angle on the same side of the transversal which opens above line m. Shade the inside of this angle.
14. The angles whose interiors you shaded in exercises 12 and 13 are called *corresponding* angles. This is because they have positions which are alike. Copy the drawing again and shade the interiors of another pair of corresponding angles.
15. How many pairs of corresponding angles are there? List them.

Now you have had some experience with the angles formed when a transversal meets two lines. Let's make sure we know the vocabulary. Angles which open between the lines are called *interior angles*. Angles *3*, *4*, *5* and *6* are interior angles.

The other angles are called *exterior angles*. Angles *1*, *2*, *7* and *8* are exterior angles.

Angles on the opposite side of the transversal that do not have the same vertex may also be called *alternate angles*. We pay particular attention to *alternate interior angles*. They are angles on opposite sides of the transversal. They are also interior angles. Furthermore, they do not have the same vertex. In the drawing on page 83, angles *3* and *6* are alternate interior angles. So are angles *4* and *5*.

Alternate exterior angles are exterior, are on opposite sides of the transversal, and have different vertices. Angles *1* and *8* are alternate exterior angles. So are angles *2* and *7*.

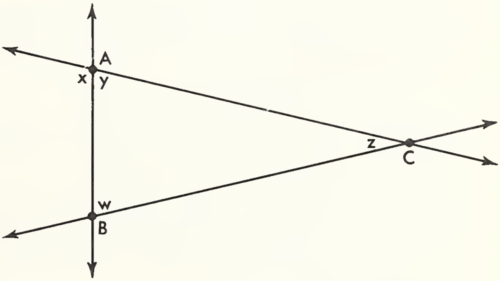
Angles which have about the same position but different vertices are called *corresponding* angles. In the drawing, angles *2* and *6* are corresponding angles. So are angles *4* and *8*. There are two more pairs of corresponding angles. They are angles *1* and *5*, and angles *3* and *7*.

PARALLEL LINES AND TRANSVERSALS

Now that we know how to speak and write about certain angles formed by lines and a transversal, we shall study parallel lines. What kinds of angles, do you suppose, are formed by parallel lines and a transversal? We shall first see what happens when the lines are not parallel, and then watch how angles change as the lines become more and more nearly parallel.

2.8 Let's Explore

1. Study this figure. Two lines, $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$, are cut by a transversal, $\overleftrightarrow{AB}$. The letters 'x', 'y', 'z' and 'w' represent the *measures* of the various angles.



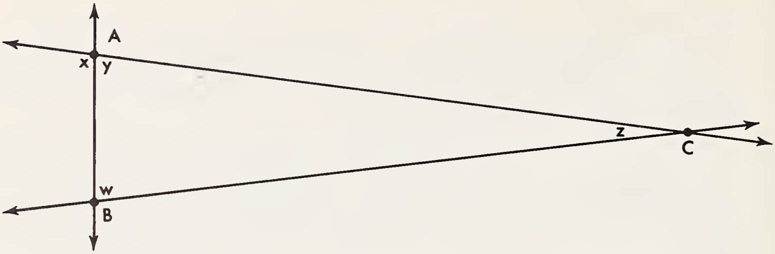
2. Remember, 'x' represents the *measure* of an angle, and 'y' represents the *measure* of an angle. Locate these angles. What is $x + y$? Why?

3. What is $y + w + z$? Why?

4. Is it true that $x + y = y + w + z$? Why?

5. Is it true that $x = w + z$? Why?

6. Now suppose we shift the position of lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ so that point C moves farther away. To help you see what happens, look at the drawing at the top of the next page.



What has happened to z, the measure of one of the angles at C?

7. Is it still true that $x = w + z$?

8. Now suppose we shift the position of lines $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ so that point C moves 10,000 miles to the right. What would happen to z, the measure of the small angle at C?

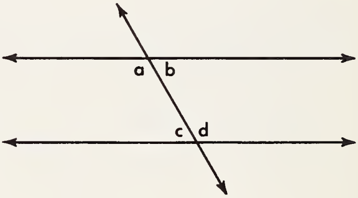
9. Would it still be true that $x = w + z$?

10. Does z get smaller as point C goes farther to the right?

11. How do x and w compare as C moves farther to the right?

12. How would x and w compare if $\overleftrightarrow{AC}$ became parallel to $\overleftrightarrow{BC}$?

You have probably discovered that if two parallel lines are cut by a transversal, the alternate interior angles are congruent. This is always true in our geometry. In this figure, the lines are parallel and $\angle a \cong \angle d$ and $\angle b \cong \angle c$.

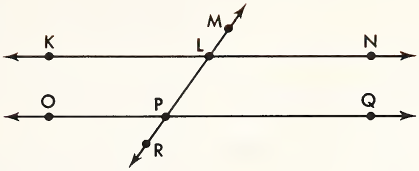


When two parallel lines are cut by a transversal, ***alternate interior angles*** *are congruent.*

There are other angle relationships when parallel lines are cut by a transversal. You will be asked to discover and prove them in the following exercises.

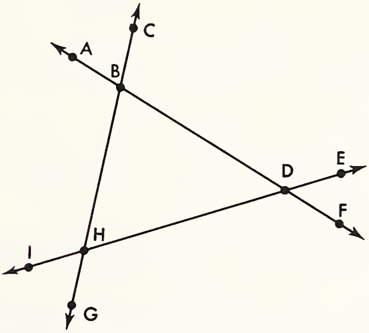
EXERCISES

1.



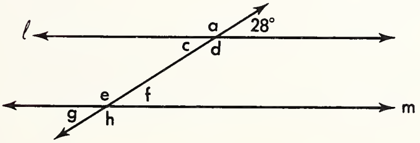
a. Name all the pairs of alternate interior angles.
b. Name all the pairs of alternate exterior angles.
c. Name all the pairs of corresponding angles.

2.



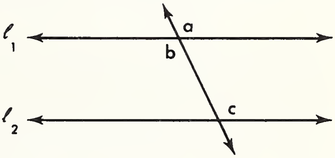
a. If you think of $\overleftrightarrow{AF}$ as a transversal, name all the pairs of alternate interior angles.
b. If you think of $\overleftrightarrow{CG}$ as a transversal, name all the pairs of alternate exterior angles.
c. If you think of $\overleftrightarrow{IE}$ as a transversal, name all the pairs of corresponding angles.

3.

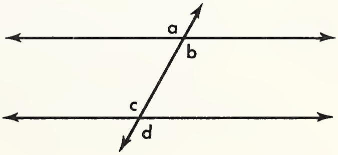


Find a, c, d, e, f, g, and h, (the measures of the angles shown) if the lines l and m are parallel.

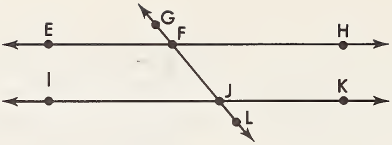
4. Here is a proof of an angle relationship. Tell why you would agree with each statement. Prove: If two parallel lines are cut by a transversal, the corresponding angles are congruent.
   a. $\angle b \cong \angle c$. Why?
   b. $\angle b \cong \angle a$. Why?
   c. $\angle a \cong \angle c$. Why?



5. Prove that if two parallel lines are cut by a transversal, the alternate exterior angles are congruent. Here is a figure you might use. You may use what was proved in exercise 4. You might also try the same kind of approach as in that exercise.



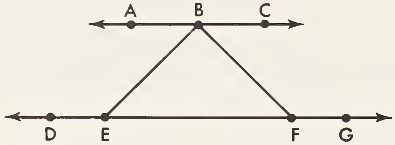
6.



a. Name as many pairs of supplementary angles on this figure as you can.

b. If you know the lines are parallel, what additional pairs of angles are supplementary?

7.



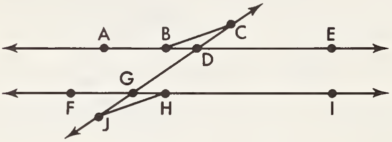
In this figure $\overleftrightarrow{AC} \parallel \overleftrightarrow{DG}$.

$m\ \angle DEB = 150°$

$m\ \angle BFG = 135°$

Find the measures of as many angles as you can.

8.



In this figure, $\overleftrightarrow{AE} \parallel \overleftrightarrow{FI}$

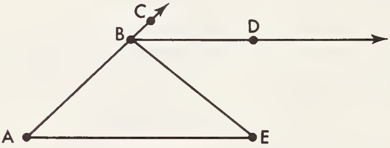
$BC \parallel JH$

$m\ \angle ADG = 40°$

$m\ \angle IHJ = 170°$

Find the measures of as many angles as you can.

9.

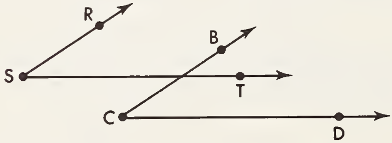


In this figure, $\overrightarrow{BD}$ bisects $\angle CBE$. $\overrightarrow{BD} \parallel \overrightarrow{AE}$

$m\ \angle CBD = 52°$

Find the measures of as many angles as you can.

10.



In this figure, $\overrightarrow{SR} \parallel \overrightarrow{CB}$

$\overrightarrow{ST} \parallel \overrightarrow{CD}$

Can you show that

$\angle RST \cong \angle BCD$?

In these exercises you should have proved two more important angle relationships when parallel lines are cut by a transversal. Let's remember them.

If two parallel lines are cut by a transversal

1. ***Alternate exterior angles*** *are congruent (or have the same measure) and*
2. ***Corresponding angles*** *are congruent (or have the same measure).*

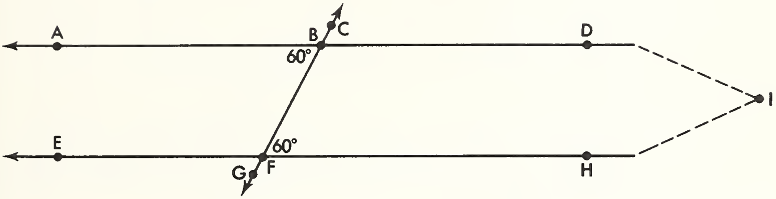
You have discovered that if two lines are parallel, then alternate interior angles are congruent. The converse of this statement is:

If alternate interior angles are congruent, then the lines are parallel.

Is the converse true?

2.9 Let's Explore

1. Here are two lines cut by a transversal, and two alternate interior angles have measure 60°.



What is the measure of $\angle FBD$? Why?

2. Now either these lines are parallel or they meet.
Suppose they meet way off on the right, at a point I.
A triangle would be formed, $\triangle BFI$. What is the sum of the measures of its angles?
3. Can you find something wrong with this? What would the measure of $\angle BIF$ be?
4. If these lines cannot form a triangle, what kind of lines must they be?

You should have discovered that the converse is true. So now we know the following, and should remember it:

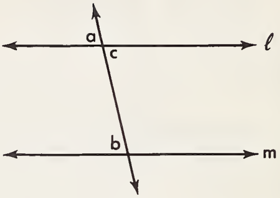
> *If two lines are cut by a transversal and alternate interior angles are congruent, then the lines are parallel.*

You also know that if two lines are parallel, then corresponding angles are congruent. The converse is:

If corresponding angles are congruent, then the lines are parallel.

Is the converse true?

2.10 Let's Explore



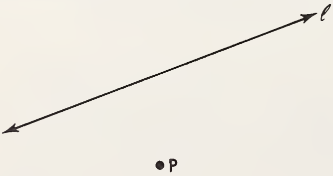
1. Here are two lines cut by a transversal. Let's suppose that a pair of corresponding angles are congruent, or in other words, $\angle a \cong \angle b$. How does a compare with c? Why?
2. Then how does b compare with c? Why?
3. What kind of angles are b and c?
4. Then if $\angle b \cong \angle c$, what do you know about lines l and m?

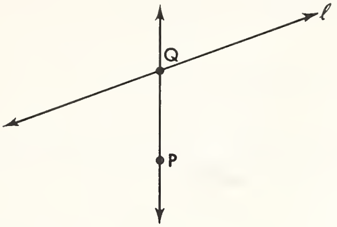
You have now shown that if corresponding angles are congruent, then lines are parallel. The converse we were wondering about is true and you should remember it:

If two lines are cut by a transversal and corresponding angles are congruent, then the lines are parallel.

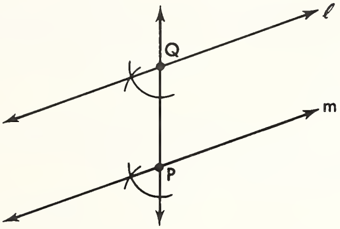
CONSTRUCTION: TO DRAW PARALLEL LINES

We can use the fact we have just proved to construct parallel lines. We do it by drawing a transversal and making corresponding angles congruent. Study this series of diagrams to see how it is done. We have given a line and a point off the line. We want to draw a parallel through the point.

Line and Point	Description
	Given line l and point P not on l.



Draw a line through P which meets l at Q. We shall use this line as a transversal.

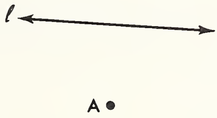


At P construct an angle congruent to the angle at Q. Draw the line m. Since corresponding angles are congruent, $l \| m$.

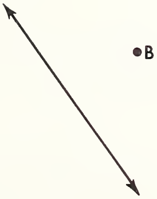
EXERCISES

1. Copy these figures. Through each point draw a line parallel to the given line. Use a compass and straightedge.

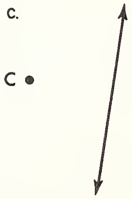
a.



b.



c.



2. Repeat question 1. Use a protractor and straightedge.

QUADRILATERALS

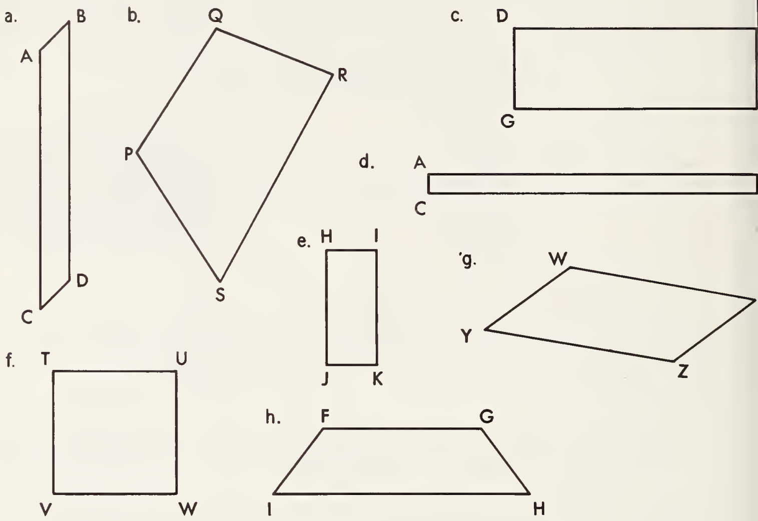
We have classified triangles according to their angles and sides. We did not do the same for quadrilaterals. Now you are ready to complete the classification of important quadrilaterals. The most important quadrilaterals are all simple, so we will not consider non-simple ones.

Remember, to classify a set of things we look for properties that will help us to distinguish one from another.

2.11 Let's Explore

Here are some quadrilaterals. Examine them and try to find some properties which will allow them to be classified.

1.



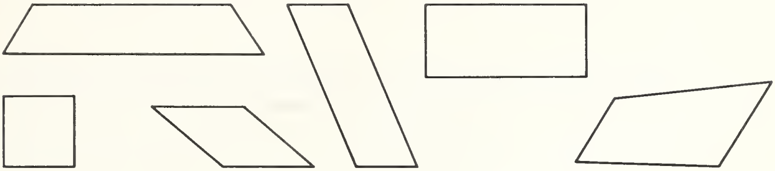
2. Do any of the quadrilaterals have a pair of parallel sides? List them.

3. Do any of the quadrilaterals have two pairs of parallel sides? List them.

You can see that classifying quadrilaterals is a little more complicated than classifying triangles. You have seen that some quadrilaterals have one or more pairs of parallel sides, and some do not. This gives us one way of classifying them. We call quadrilaterals having at least one pair of parallel sides *trapezoids*.

A simple quadrilateral that has one or more pairs of parallel sides is called a ***trapezoid.***

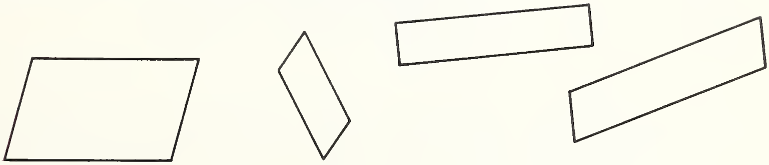
Here are a few examples of trapezoids. Can you tell which pairs of sides are parallel?



Some trapezoids have more than one pair of parallel sides. They are important and are called *parallelograms.*

A trapezoid that has two pairs of parallel sides is called a ***parallelogram.***

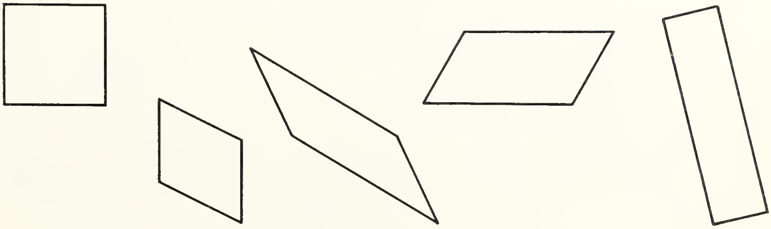
Here are a few examples of parallelograms.



Parallelograms have many interesting properties. See how many you can discover.

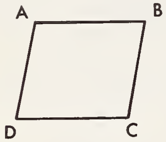
2.12 Let's Explore

Here are some parallelograms. Study them carefully.



1. Measure the sides of the parallelograms at the bottom of page 93 to see what you can discover about them.
2. Measure the angles to see what you can discover about them.

You have probably discovered that parallelograms have some sides congruent. In some cases all four sides are congruent. In every case the opposite sides are congruent. In this picture, $\overline{AB}$ and $\overline{DC}$ are opposite sides, and they are congruent. Can you find another pair of opposite sides? Are they congruent? You may prove this later when you take a formal geometry course.



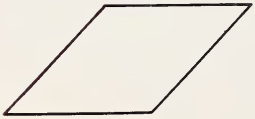
The angles of a parallelogram also have a relationship. *Opposite angles* are congruent. Angles A and C are opposite angles, and are congruent. Can you find another pair of opposite angles?

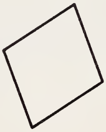
In any parallelogram
1. Opposite sides are congruent.
2. Opposite angles are congruent.

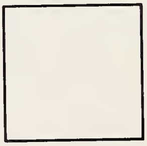
In your study of properties of parallelograms, you probably discovered that some parallelograms have all four sides congruent. Also, some parallelograms had right angles. These two properties give us special kinds of parallelograms.

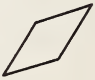
If a parallelogram has four sides congruent, then it is called a ***rhombus.*** (*rhombi, plural*)

Here are some examples of rhombi.



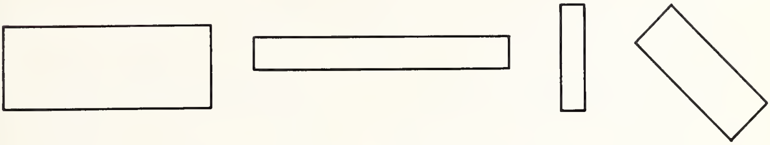




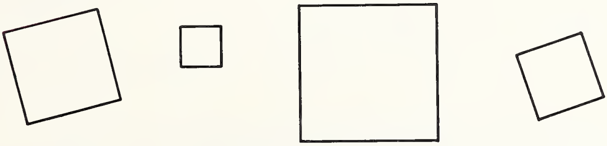


If a parallelogram has four right angles, then it is called a ***rectangle.***

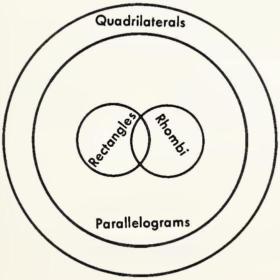
You are familiar with rectangles. Here are a few.



If a parallelogram has four congruent sides and has four
right angles, then it is called a ***square.***
If a rectangle has four congruent sides,
then it is called a ***square.***
If a rhombus has four right angles,
then it is called a ***square.***

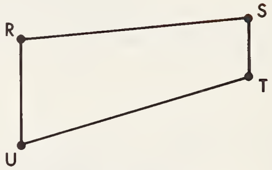


To help you keep track of this classification, study this Venn diagram. Notice the set of parallelograms is a subset of the set of quadrilaterals. This shows that all parallelograms are quadrilaterals. Are all rectangles parallelograms? Are all rectangles quadrilaterals? Are all rhombi parallelograms? Are all rhombi quadrilaterals? Are all rhombi rectangles? The sets of rectangles and rhombi intersect. This shows that some rectangles are also rhombi. What are they called? Where should the set of trapezoids be shown?

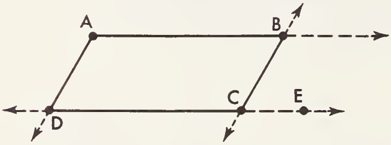


EXERCISES

1. a. Copy the Venn diagram on page 95.
   b. Label the portion for squares.
   c. Draw and label a portion for trapezoids.
   d. Draw and label a portion for polygons.
   e. Draw and label a portion for simple closed curves.
2. What is a rectangle that is a rhombus called?
3. What is a rhombus that is a rectangle called?
4. Here is a quadrilateral.
   a. Name the pairs of opposite angles.
   b. Name the pairs of opposite sides.

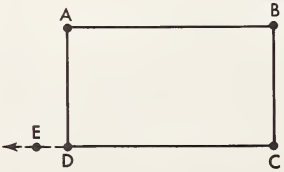


5. Here is a proof that opposite angles of a parallelogram are congruent. Tell why each statement is acceptable.



   a. $\overline{AD} \parallel \overline{BC}$
   b. $\angle ADC \cong \angle BCE$
   c. $\overline{DC} \parallel \overline{AB}$
   d. $\angle BCE \cong \angle ABC$
   e. $\angle ADC \cong \angle ABC$

6. Another possible definition for a rectangle is:
   A parallelogram with one right angle is called a *rectangle*.
   Show that if a parallelogram has one right angle, then the other three angles must be right angles. Suppose $ABCD$ is a parallelogram and that $\angle B$ is a right angle. Here is a list of statements for the proof. You supply the reasons why each statement is acceptable.

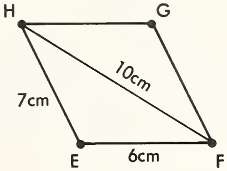


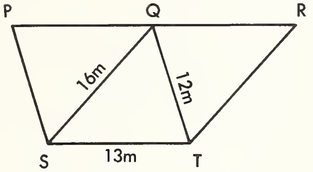
   a. $\angle ADC$ is a right angle.
   b. $\angle ADE$ is a right angle.
   c. $\angle DAB$ is a right angle.
   d. $\angle BCD$ is a right angle.

   Does this prove that the new definition is just as good as the one we made originally?

7. Find the perimeter of
   a. a parallelogram if one side is 8.6″ long and another is 3.8″ long.
   b. a rectangle if its length is $9\frac{3}{4}$″ and its width is $7\frac{1}{3}$″.
   c. a rhombus if a side is 5.9″ long.
   d. a square if a side is $26\frac{1}{3}$ cm. long.

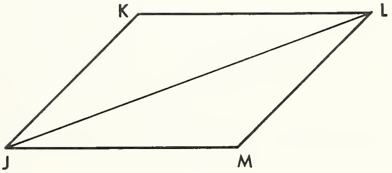
8. Quadrilateral $EFGH$ below is a parallelogram. Find its perimeter using the given measures.



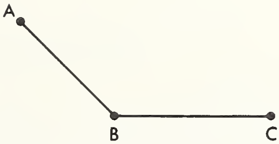


9. The measures of the sides of $\triangle SQT$ above are given. Quadrilateral $PQTS$ is a parallelogram. Quadrilateral $QRTS$ is a parallelogram.
   a. Find the perimeter of quadrilateral $PRTS$.
   b. What kind of a quadrilateral is it?

10. $m\ \angle JLM = 35°$.
    $m\ \angle LMJ = 135°$.
    $\overline{KL} \parallel \overline{JM}$.
    $\overline{KJ} \parallel \overline{LM}$.
    Find the measures of as many angles as you can.



11. It is true that if the opposite sides of a quadrilateral are congruent, the quadrilateral is a parallelogram.
    a. Copy this figure.
    b. Using the above statement, try to find an easy way to construct a parallelogram on the figure.



12. Draw a parallelogram with one side $1\frac{1}{2}$″ long and another side $\frac{1}{2}$″ long.
13. Draw a rectangle with a 3″ length and a $2\frac{1}{2}$″ width.
14. Draw a rhombus with a side 1″ long and an angle with measure 40°.
15. Draw a square with one side 3″ long.

Review Practice

1. Write expanded numerals for each of these whole numbers. Use zero exponents.

Example:

$$356 = 3 \cdot 10^2 + 5 \cdot 10^1 + 6 \cdot 10^0$$

a. 925
b. 2813
c. 49,832
d. 75,327

2. Write expanded numerals for each of these numbers. Use the base indicated.

a. 324_7
b. 4123_5
c. 2132_4
d. $72{,}103_8$

3. Find the decimal numeral for each number of exercise 2.

4. Write expanded numerals for each of these numbers of arithmetic.

Example:

$$27.35 = 2 \cdot 10^1 + 7 \cdot 10^0 + 1 \cdot (\tfrac{1}{10})^1 + 5 \cdot (\tfrac{1}{10})^2$$

a. 32.42
b. 123.403
c. 181.00005
d. 3.00000008

CIRCLES

Most of the plane figures you have studied have been made up of lines or parts of lines. We shall now study a different kind of plane figure.

2.13 Let's Explore

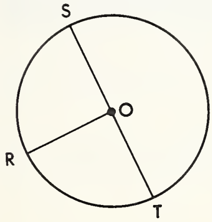
This exercise is to be done in the plane of your paper.

1. a. Draw a point, O, in about the middle of your paper.
   b. Using your ruler find a point 2″ from O. Label it 'A'.
   c. Find another point 2″ from O. Label it 'B'.
   d. Find 10 more points 2″ from O.
   e. Put the point of your compass on O and draw the set of all points 2″ from O.
   f. Does O belong to the set?

2. a. Draw a point S, in about the middle of your paper.
   b. Using your compass, draw the set of all points that are 3 centimeters from S.
   c. Draw a segment from S to some point of that set.
   d. What is the measure of this segment?
   e. What is the measure of any segment from S to a point of the set?

3. a. Draw a point T, in about the middle of your paper.
   b. Using your compass, draw the set of all points that are $1\frac{1}{2}$ inches from T.
   c. Choose any two points of that set. Use your straightedge to draw a segment between them.
   d. Which points of your set belong to that segment?
   e. Through T, draw a segment with endpoints on the set.
   f. Is this segment the longest one you could draw between two points of the set?
   g. What is its measure?

You have probably discovered that the set of points in a plane 2″ from a given point does not look like a line, or a segment or a polygon. This set of points is called a *circle*. A circle is a plane figure. It is determined by two things; a given point and a given distance. The point is called the *center* of the circle. It is not a part of the circle, but it helps determine the circle. The given distance will be the measure of all segments from the center of the circle to a point of the circle. Each of these segments is called a *radius* (plural, *radii*). The given distance is the measure of every radius. In the picture O is the center of the circle. R is a point of the circle and $\overline{OR}$ is a radius. The segments $\overline{OS}$ and $\overline{OT}$ are also radii.



Another kind of segment associated with a circle is called a *diameter*. A diameter is a segment which contains the center and whose endpoints are on the circle. The segment $\overline{ST}$ is a diameter. Do all diameters of a circle have the same measure? How does the length of a radius compare with the length of a diameter?

A circle may be drawn with a compass. Since a compass keeps distances, a circle can be made by putting the metal tip on the center and having the compass set at the distance needed. You rotate the compass about the metal tip, and the pencil draws the circle.

A ***circle*** *is a set of points on a plane which are a certain distance from a given point, called the* ***center.***
A ***radius*** *is a segment whose endpoints are the center and a point of the circle.*
A ***diameter*** *is a segment containing the center and whose endpoints are on the circle.*

When you studied polygons you learned to find perimeters. The perimeter of a polygon is the sum of the measures of its sides. It is also the "distance around" the polygon. A circle does not have sides, so we could not find the sum of the measures of the sides. We might, however, think about the "distance around" it.

2.14 Let's Explore

1. a. On stiff paper draw a circle whose diameters are 2″ long.
   b. Carefully place a string around the circle. Straighten out the string and measure its length.
   c. Check your work another way. Cut out the circle to make a disc. Make a mark on the disc. Roll it along a line drawn on your piece of paper, making a mark where it starts. Roll it completely around once. Make a mark where it stops. Measure the distance between the marks.
   d. Divide the distance around the circle by the length of a diameter.
2. Draw a circle whose diameters are 3″ long. Repeat exercise 1 for this circle.
3. Draw a circle whose diameters are 5″ long. Repeat exercise 1 for this circle.
4. How does the distance around a circle compare with the length of a diameter?

The distance around a circle is called its *circumference.* You have probably discovered that the circumference of a circle is a little more than three times the length of a diameter. Although you did not prove it, this is true for all circles.

If you multiply the measure of the diameter by a certain number "a little more than three" you would get the circumference. It is hard to tell what this certain number "a little more than three" is, because this number does not belong to any set of numbers that you have

Francois Viète

1540-1603

Viète was a French lawyer who dabbled in mathematics as a recreation. Symbolism varied widely from mathematician to mathematician. In his works, Viète used consonants to represent numbers that he knew and vowels as variables. He used '•' for '+' and '=' for '−'. He was interested in methods for solving number sentences.

During a war with Spain, Viète broke a Spanish code of several hundred characters. Was it magic or mathematics? Today, mathematics plays an important part in breaking codes.

Viète was the first mathematician to find a way of calculating π that did not involve measuring a circle. This proved that π was a number not limited to situations involving the circle.

Jules Henri Poincaré

1854-1912

The Frenchman Poincaré was a prolific writer, having composed over 1500 essays and numerous books on every known branch of mathematics. At first laughing at some of the *new* geometries, he later turned his attention to them and found a model in which all of the unusual properties of the geometry of Lobachevsky actually existed. This model demonstrated that these new geometries were not merely pleasant exercises in logic, but could be applied to the world, which, after all, does look very Euclidean.

developed. It belongs to another set of numbers. You will study them later. But you do know that it is a little more than 3. This number has no fractional numeral. Its decimal numeral does not terminate or repeat. Rounded to ten decimal places it is 3.1415926535. Because its decimal numeral does not end and there is no fractional numeral for it, we name it in a different way. We use the Greek letter 'π', called "pi", to name it. This number troubled the ancient Greeks when they studied the circle, because it has no fractional numeral. Archimedes proved that $3\frac{1}{7} > \pi > 3\frac{10}{71}$. We use an arithmetic number that is close to π when we calculate. Since $\pi \approx 3.14$ we often use 3.14 for π.

We sometimes also use $\frac{22}{7}$ or $3\frac{1}{7}$. However, $3.14 < \pi$, while $\frac{22}{7} > \pi$. If we are using decimal numerals we would probably use 3.14 for π. If we are using fractional numerals, we would be more apt to use $\frac{22}{7}$.

CIRCLE FORMULAS

You have already seen that if we divide the circumference of a circle by the length of a diameter, the result is π. This is the definition of π. We can say this with the number sentence

$$\pi = \frac{C}{d}$$

where 'C' represents the circumference, and 'd' represents the length of a diameter.

Remember that multiplication and division are opposite operations. If we divide C by d we get π. If we then multiply the quotient by d, we get C, the number with which we started. We can say this with the number sentence

$$C = \pi \cdot d.$$

This sentence tells us that if we multiply the length of a diameter by π we get the circumference. We shall very often use this sentence as a formula for finding circumference.

Example:

What is the circumference of a circle whose diameters measure 5 inches?

$C = \pi \cdot d \approx 3.14 \cdot 5$

$C \approx 15.7$. The circumference is about 15.7″.

Suppose we do not know the length of a diameter, but know the length of a radius. How could we find the circumference? You know that a diameter is twice as long as a radius. We could say this with the number sentence

$$d = 2 \cdot r,$$

where 'd' represents the length of a diameter and 'r' represents the length of a radius. Now let's substitute '$2 \cdot r$' for 'd' in our other formula. We get

$$C = \pi \cdot (2 \cdot r),$$

or, using the commutative and associative properties,

$$C = 2 \cdot \pi \cdot r.$$

We can use this last number sentence as a formula. With it we can find circumference when we know the length of a radius.

Example:

What is the circumference of a circle if the length of a radius is 9 ft?

$$C = 2 \cdot \pi \cdot r \approx 2 \cdot \tfrac{22}{7} \cdot 9$$
$$C \approx \tfrac{396}{7} \text{ or } 56\tfrac{4}{7}$$

The circumference is about $56\frac{4}{7}$ ft.

Be sure to remember these circle formulas.

In any circle, if the length of a radius is **r**, *the length of a diameter is* **d** *and the circumference is* **C**, *then*

$$\mathbf{d} = 2 \cdot \mathbf{r},$$
$$\mathbf{C} = \pi \cdot \mathbf{d}, \text{ and}$$
$$\mathbf{C} = 2 \cdot \pi \cdot \mathbf{r}$$

EXERCISES

1. a. Using a compass, draw a circle with radius 4 cm long.
   b. Draw and label the center, a radius and a diameter.
2. Draw a circle whose diameter is:
   a. 2″ long b. 6 cm long c. 3″ long
3. Draw a circle whose radius is $\frac{3}{4}$″ long. Draw two radii which are perpendicular.
4. a. Draw a circle whose diameter is 3″ long.
   b. Into how many sets does the circle separate the plane?
   c. Shade them each differently.

5. Draw two circles whose intersection is:
   a. the empty set.
   b. 1 point.
   c. 2 points.
   d. 3 points.

6. a. Study this definition:
   Two or more circles in a plane that have the same center, but have radii of different measures, are called *concentric circles.*
   b. Draw three concentric circles.

7. a. Draw a segment at least an inch long.
   b. Using only a compass and straightedge (do not measure), draw a circle that contains the endpoints of the segment and whose center is on the segment.

8. Find the circumference of a circle whose diameter measures:

a. 6″	d. $\frac{7}{5}$ in.
b. 1 cm.	e. 7.1 m.
c. 5280 ft.	

9. Find the diameter of a circle whose circumference is:

a. 6.28 cm.	d. $7\frac{1}{3}$
b. 3140 miles	e. 11 ft.
c. 188.4 in.	

10. Find the circumference of a circle whose radius measures:

a. 1 ft.	d. 7.8 yds.
b. 20 cm.	e. 10.2 m.
c. $\frac{3}{4}$ in.	

11. Find the radius of a circle whose circumference is 188.4 in.

12. Here is a plane figure. It is called a *semi-circle.* It is half a circle. Its measure is $\frac{22}{35}$″. What is the radius of the semi-circle?

13. Find the circumference of a circle whose radius measures:
   a. 3″ b. 6″ c. 12″ d. 24″
      i. What happens to the circumference if the length of the radius is doubled?
      ii. What happens to the circumference of a circle if the length of the radius is tripled? Back up your conjecture with some evidence.

DO YOU KNOW THESE TERMS?

To show that you do know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

figure (59)
segment (60)
ray (60)
perpendicular (61)
congruent (63)
straightedge (65)
compass (65)
construction (65)
bisect (68)
angle bisector (68)
midpoint (74)
perpendicular bisector (74)
parallel lines (81)
transversal (82)
skew line (82)
interior angles (84)
corresponding angles (84)
exterior angles (84)
alternate interior angles (84)
alternate exterior angles (84)
quadrilateral (92)
trapezoid (92)
parallelogram (93)
rhombus (94)
rectangle (95)
square (95)
circle (99)
radius (99)
diameter (99)
circumference (100)
pi (101)
concentric circles (103)
semi-circle (103)

CHAPTER SUMMARY

Geometric figures (59)
Compass and straight edge (65)
Construction: To copy an angle (66)
Construction: To bisect an angle (68)
Construction: To bisect a segment (74)
Construction: To draw a line perpendicular to a line at a point on a line (78)
Construction: To draw a perpendicular to a line from a point not on the line (78)
Parallel lines (81)
Transversals (82)
Construction: To draw parallel lines (90)
Quadrilaterals (91)
Circles (98)
Circle formulas (101)

REVIEW EXERCISES

1. For what purpose is a straightedge used?

2. For what purpose is a compass used?

3. Draw examples of each of the following:
   a. A right angle.
   b. An acute angle.
   c. An obtuse angle.
   d. A straight angle.
   e. Two adjacent supplementary angles.
   f. Two non-adjacent supplementary angles.
   g. Two adjacent complementary angles.
   h. Two non-adjacent complementary angles.
   i. Vertical angles.

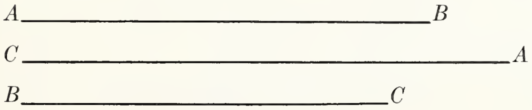
4. Use a compass and straightedge to copy the angles you drew in exercise 3-a, 3-b, and 3-c.

5. a. Draw an obtuse triangle.
   b. Use your compass and straightedge to construct an altitude of your triangle.

6. a. Draw an acute triangle.
   b. Use your compass and straightedge to construct an altitude of your triangle.

7. a. Draw an isosceles triangle.
   b. Bisect the angle with a vertex where the two sides of equal measure intersect.
   c. Extend the angle bisector till it intersects the opposite side of the triangle.
   d. Check to see if your construction was correct by measuring to see if the angle bisector also bisected the side of the triangle. (This would not happen in a scalene triangle.)

8. Construct a triangle whose sides have lengths of these segments.

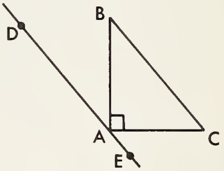


9. Bisect each of the sides of the triangle you drew for exercise 8. (If your constructions are correct the bisectors will meet at a point.)

10. a. Draw a line segment AB, of any length.
    b. Construct the perpendicular bisector of the segment.

- c. Using the point where the bisector intersects the segment (D) as one end point, construct on the bisector a segment half as long as the original segment. Label the other end C.
- d. From C draw a segment with an endpoint at A and a segment with an endpoint at B.
- e. Tell whether the triangle you have drawn is scalene, isosceles or equilateral.
- f. Tell whether the triangle is right, obtuse or acute.

11. ABC is a right scalene triangle.
$\angle BAC$ is a right angle.
$\overline{BC} \parallel \overline{DE}$
$m\ \angle ABC = 50°$



Find the measure of the following angles:
- a. $\angle BCA$
- b. $\angle DAB$
- c. $\angle CAE$

12. a. Construct an equilateral triangle.
- b. Through one vertex construct a segment parallel to the opposite side of the triangle.

13. Tell which of the following are true:
- a. All squares are rhombi.
- b. All rhombi are squares.
- c. All squares are rectangles.
- d. All rectangles are squares.
- e. Some rectangles are squares.
- f. All trapezoids are parallelograms.
- g. All quadrilaterals are trapezoids.
- h. All parallelograms are trapezoids.
- i. All squares are trapezoids.
- j. All rhombi are parallelograms.
- k. All quadrilaterals are parallelograms.
- l. Some quadrilaterals are hexagons.
- m. Opposite sides of a quadrilateral have equal measure.
- n. Opposite sides of a parallelogram have equal measure.
- o. A regular quadrilateral is called a rectangle.
- p. Opposite angles of a parallelogram have equal measure.
- q. A parallelogram with adjacent sides of equal measure is called a rhombus.
- r. A parallelogram with adjacent angles of equal measure is called isosceles.
- s. A parallelogram with four right angles is a square.
- t. Some rectangles are rhombi.

14. Draw an example of a convex quadrilateral and show that it is convex.

15. Draw an example of a concave quadrilateral and show that it is concave.

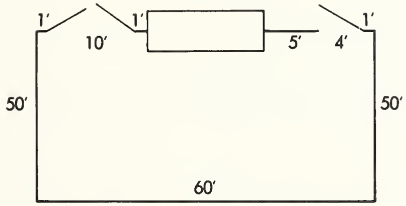
16. What is the name of a line segment through the center of a circle, with endpoints on the circle?

17. What is the name of a line segment with one endpoint on the circle and the other at the center of the circle?

18. Find the circumference of a circle with a diameter whose measure is 4 centimeters.

19. Find the circumference of a circle with a radius whose measure is 2″.

20. Mr. Self wished to put a fence around the back part of his lot as shown in the sketch. The fencing will cost $1.00 per foot, not counting the gates. The posts cost $1.10 each and are to be no more than 10 feet apart. The larger double gate costs $30 and the smaller gate costs $15.



a. Find the cost of installing the fence.

b. If he pays for the fence over a period of five years and the company charges 6% of the purchase price for carrying charges, how much must be added to the price he must pay? (Note: This is 6% of the purchase price. The 6% here does not mean 6% per year as with interest.)

c. Add the carrying charges to the purchase price to find the total cost.

d. Divide by the number of months in five years to find the amount of his monthly payments.

21. a. Find the sale price of each of the following items at a 15% discount sale:

	item		marked price
i.	a 4-piece bedroom suite		$250.00
ii.	a mattress		99.50
iii.	a box spring		99.50
iv.	3 scatter rugs	each	10.00

b. Find the total bill for the above purchases at the sale, including a 2% sales tax.

22. The salesman who made the sale described in exercise 28 receives a 5% commission on all sales. What was his commission on that sale?

23. Copy and complete:

a. 5 lb = ________ oz
b. 10 cm = ________ m
c. 15 grams = ________ mg
d. 3 hr = ________ days
e. 3 sq ft = ________ sq in

24. For what integers will each of the following be true?

a. $x > {}^{+}5$
b. $x \leq {}^{-}3$
c. $a \neq 0$
d. $|a| \not> 0$
e. $x + {}^{-}3 = {}^{+}2$

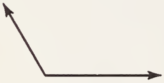
25. Study the conditional sentence, "If your score was above 90, then your grade was an A."

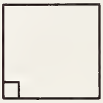
a. What is the antecedent of the sentence?
b. What is the consequent of the sentence?
c. Write the converse of the sentence.

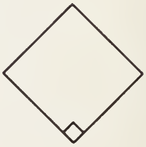
26. $A = \{2, 4, 6, 8, 10, 12, \cdots\}$
$B = \{1, 2, 3, 4, 5, 6, \cdots\}$
Which of the following are true?

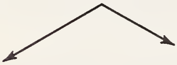
a. $A \subset B$
b. $B \subset A$
c. $A = B$
d. $A \cup B = A$
e. $A \cup B = B$
f. $A \cap B = \phi$
g. $A \cap B = A$
h. $A \cap B = B$
i. $A \cup B = \phi$
j. $\phi = A$

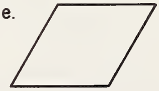
27. Which of the following are congruent figures?

a.


b.


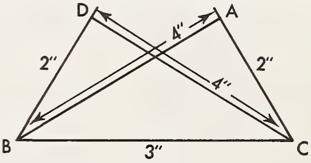
c.


d.


e.


28. Which of the sentences about the following figure are true?

a. $AB = CD$
b. $\overline{AB} = \overline{CD}$
c. $AC \cong BD$
d. $AB \cong CD$
e. $\overline{AB} \cong \overline{CD}$



29. Draw a Venn diagram to show the intersection of the set of multiples of 5 between 20 and 50 and the set of factors of 100.

30. Tell whether 346,896 is divisible by each of the following:
    a. 2
    b. 10
    c. 5
    d. 9
    e. 3

READING LIST

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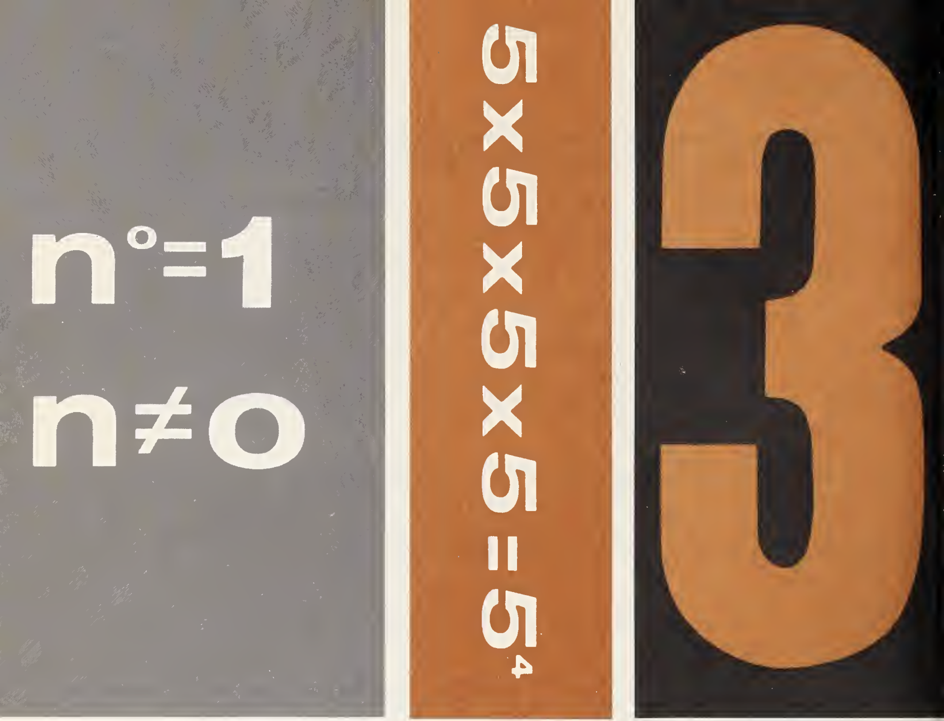
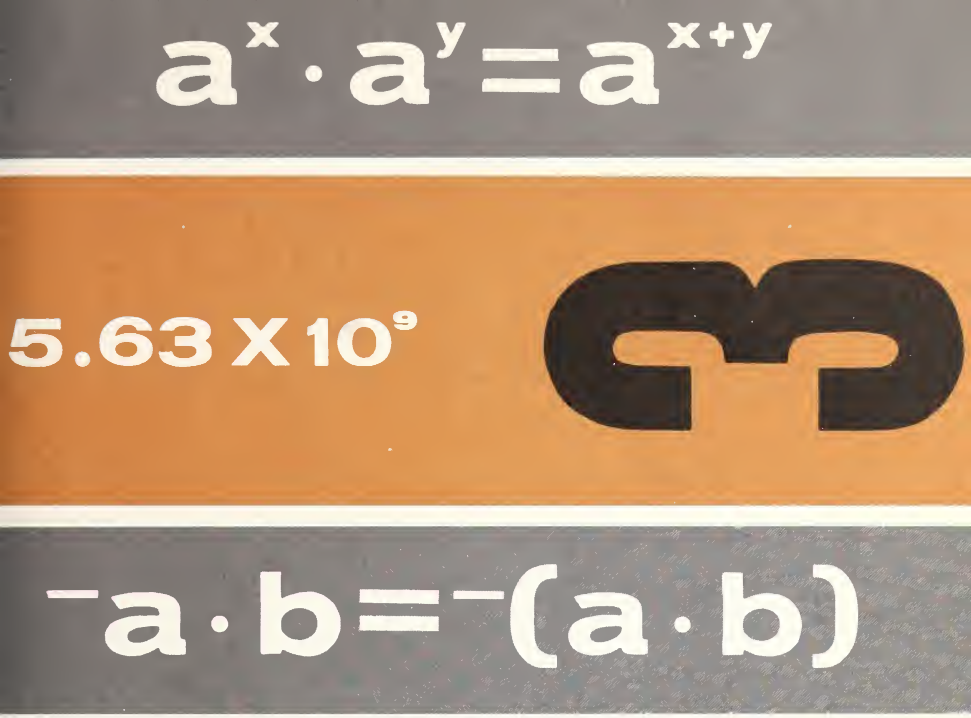


CHAPTER THREE

Exponents and Scientific Notation



POWERS OF NUMBERS

Two names for the number 125 are '$5 \times 5 \times 5$' and '5^3'. Another name for 125 is 'the third power of 5.' Thus '5^3' may also be read "the third power of 5," or "five to the third power." The numeral '5^3' consists of two parts—the '5' and the superscript, '^{3}'. The numbers represented by the '5' and the '^{3}' are called the *base* and the *exponent*, respectively. The exponent tells us how many times the base is used as a factor unless it is 1 or 0. Perhaps you remember that 'a^1' means the same as 'a'. We defined the exponent 1 this way. You may also remember that we defined 'a^0' to represent a symbol for the number 1, whenever a is not zero.

When we factor 5^3 we may write '$5 \times 5 \times 5$'. When we write the numeral '5^3' for the third power of 5, we say we are writing *exponential notation* for 125.

The exercises on the following page will give you practice in changing from exponential notation to decimal notation and from decimal to exponential notation.

EXERCISES

1. Write decimal numerals for these numbers:
   - a. 2^3
   - b. 3^2
   - c. 2^5
   - d. 2^2
   - e. 9^2
   - f. 5^3
   - g. 10^4
   - h. 3^3
   - i. 3^4
   - j. 6^2
2. Write exponential notation for these numbers:
   - a. 9
   - b. 8
   - c. 25
   - d. 1000
   - e. 125
   - f. 64
   - g. 27
   - h. 144

MULTIPLYING POWERS

You know how to multiply numbers of arithmetic using decimal numerals or fractional numerals. What kind of patterns can we find, so that we can learn to use exponential notation for multiplying?

3.1 Let's Explore

1. Write exponential notation for $8 \times 8 \times 8 \times 8$.
2. Factor 3^2.
3. Factor 3^3.
4. Factor $3^2 \cdot 3^3$.
5. Write another numeral, using a single exponent symbol, for the number of exercise 4.
6. What does '2^5' mean? (Remember that the exponent tells us how many times the base is used as a factor.)
7. What does '2^3' mean?
8. What does '$2^5 \cdot 2^3$' mean?
9. Using a single exponent symbol, write another numeral for $2^5 \cdot 2^3$.
10. Factor 5^4.
11. Factor 5^6.
12. Factor $5^4 \cdot 5^6$.

13. Write another numeral, using a single exponent symbol, for the number of exercise 12.
14. Write exponential notation for the product $5^{10} \cdot 5^{12}$. (Factor first, then write exponential notation.)
15. Write exponential notation for the product $10^{20} \cdot 10^{5}$. (Remember what the exponent tells us about the base.)
16. Write exponential notation for the product $3^7 \cdot 3^2 \cdot 3^5$.
17. What does 'a^3' mean?
18. What does 'a^2' mean?
19. Factor $a^3 \cdot a^2$.
20. Write exponential notation for $a^3 \cdot a^2$.
21. Write another numeral for $5^2 \cdot 3^3$.
22. How would you write exponential notation for the number of exercise 21?
23. Explain why exercises 15 and 21 are not alike.
24. Write exponential notation for the product $5^3 \cdot 5^1$.
25. Write exponential notation for the product $7^4 \cdot 7^0$.

You have probably discovered a way to multiply using exponential notation. If the bases are alike, you use that base and the sum of the exponents. Why does this work? In this example, we can explain it. '$3^2 \cdot 3^4$' means that 3 is used as a factor twice and then four more times. Altogether it is used as a factor 6 times. Therefore,

$$3^2 \cdot 3^4 = 3^{2+4}, \text{ or } 3^6.$$

If an exponent is 1 the exponent does not tell how many times the base is used as a factor, because in a product we must have at least *two* factors. How do we explain that exponents are still to be added if one or more exponents are 1?

'$5^6 \cdot 5^1$' means what?

By agreement, '5^1' means 5. Now '5^6' means $5 \times 5 \times 5 \times 5 \times 5 \times 5$. So '$5^6 \cdot 5^1$' means

$$(5 \times 5 \times 5 \times 5 \times 5 \times 5) \times 5.$$

Altogether 5 is used as a factor 7 times. Therefore

$$5^6 \cdot 5^1 = 5^{6+1}, \text{ or } 5^7.$$

Remember the meaning of the zero exponent, and let's see that the rule of adding exponents still works when one or more exponents are zero.

'$3^4 \cdot 3^0$' means what?

By agreement, '3^0' means 1. Now '3^4' means $3 \times 3 \times 3 \times 3$, so '$3^4 \cdot 3^0$' means

$$(3 \times 3 \times 3 \times 3) \times 1, \text{ or simply}$$

$$3 \times 3 \times 3 \times 3.$$

Altogether 3 is used as a factor 4 times. Therefore

$$3^4 \cdot 3^0 = 3^{4+0}, \text{ or } 3^4.$$

For any non-zero number ***a*** *and any whole numbers* ***x*** *and* ***y***,

$$a^x \cdot a^y = a^{x+y}.$$

EXERCISES

Multiply. Write the simplest exponential notation for each answer.

1. $3^2 \times 3^3$
2. $5^4 \cdot 5^2$
3. $4^8 \times 4^3$
4. $10^3 \cdot 10^5$
5. $2^3 \times 2^4 \times 2^2$
6. $6^2 \cdot 6^4$
7. $3^5 \times 3$
8. $4^7 \cdot 4^1$
9. $5^{17} \times 5 \times 5^3$
10. $10^6 \cdot 10^2 \cdot 10$
11. $7^5 \times 7 \times 7^0$
12. $3^0 \cdot 3^0 \cdot 3^0$
13. $12^1 \times 12^3 \times 12^5$
14. $6^7 \cdot 6^5 \cdot 6^0 \cdot 6$
15. $2^7 \times 2^0 \times 2^6 \times 2^4$
16. $5^4 \cdot 5^1 \cdot 5^7 \cdot 5^0$
17. $3^4 \times 3^5 \times 7^2$
18. $8^7 \cdot 7^8 \cdot 8^0 \cdot 7^1$
19. $7^0 \times 5^4 \times 6^0$
20. $3^6 \cdot 7^4 \cdot 3^2 \cdot 7^1$

Review Practice

1. What is the additive inverse of each of these integers?
   a. 4
   b. $^-5$
   c. 12
   d. $^-7$
   e. 0
   f. $(2 + 4)$
   g. $^-(^-7)$
   h. $(5 - 2)$
   i. $^-(^-(^-5))$

2. Add:
   - a. $17 + 14$
   - b. $13 + {}^{-}6$
   - c. ${}^{-}5 + {}^{-}31$
   - d. ${}^{-}45 + 16$
   - e. ${}^{-}75 + {}^{-}142$
   - f. $631 + {}^{-}941$
   - g. $356 + 712$
   - h. ${}^{-}457 + 235$

3. To subtract an integer b from an integer a, we ______ the ______ ______ of b to a.

4. Subtract:
   - a. $14 - 2$
   - b. $13 - {}^{-}5$
   - c. ${}^{-}12 - 17$
   - d. ${}^{-}6 - 41$
   - e. ${}^{-}14 - {}^{-}12$
   - f. $146 - {}^{-}73$
   - g. ${}^{-}456 - 702$
   - h. ${}^{-}981 - {}^{-}653$

5. What is the reciprocal (multiplicative inverse) of each of these numbers of arithmetic?
   - a. 5
   - b. $\frac{1}{3}$
   - c. $\frac{3}{4}$
   - d. $\frac{1}{12}$
   - e. $\dfrac{1}{\frac{1}{5}}$
   - f. $\dfrac{1}{\frac{3}{4}}$
   - g. $\frac{5}{8}$
   - h. $\frac{9}{10}$

6. Multiply and simplify:
   - a. $\frac{2}{3} \cdot \frac{9}{5} \cdot \frac{7}{8}$
   - b. $\frac{5}{9} \cdot \frac{3}{2} \cdot \frac{8}{5}$
   - c. $\frac{7}{2} \cdot \frac{3}{8} \cdot \frac{4}{7}$
   - d. $\frac{12}{7} \cdot \frac{16}{3} \cdot \frac{14}{5}$
   - e. $\frac{25}{12} \cdot \frac{19}{10} \cdot \frac{6}{17}$
   - f. $\frac{45}{7} \cdot \frac{3}{5} \cdot \frac{14}{9}$
   - g. $\frac{125}{21} \cdot \frac{18}{15} \cdot \frac{7}{5}$
   - h. $\frac{360}{4} \cdot \frac{64}{3} \cdot \frac{85}{18}$

7. To divide an arithmetic number x by a number y, we can ______ x by the ______ of y.

8. Which of these sentences are true?
   - a. $\frac{3}{4} = 3 \cdot \frac{1}{4}$
   - b. $\frac{6}{7} = \frac{1}{7} \cdot \frac{1}{6}$
   - c. $\frac{9}{5} = \frac{1}{9} \cdot 5$
   - d. $\frac{4}{7} = \frac{1}{7} \cdot 4$
   - e. $\dfrac{1}{\frac{5}{4}} = \frac{4}{5}$
   - f. $\dfrac{1}{\frac{3}{8}} = \frac{3}{8}$
   - g. $\dfrac{1}{\frac{1}{2}} = 2$
   - h. $5 = \dfrac{1}{\frac{1}{5}}$

9. Divide:
   - a. $\frac{3}{2} \div 4$
   - b. $13 \div 7$
   - c. $3 \div \frac{1}{2}$
   - d. $\frac{3}{8} \div \frac{5}{4}$
   - e. $6 \div 5$
   - f. $\frac{12}{7} \div 5$

DIVIDING USING EXPONENTIAL NOTATION

You have practiced multiplying using exponential notation. It is simple to do. We merely add exponents if bases are alike. Can we find an easy way to divide?

3.2 Let's Explore

1. Consider the fractional phrase $\frac{‘2^3’}{2^2}$.
   a. Factor the numerator and denominator.
   b. Simplify.
   c. Since $\frac{‘2^3’}{2^2}$ is another name for $2^3 \div 2^2$, what is the simplest name for $2^3 \div 2^2$?
2. Simplify $\frac{3^5}{3^2}$ by factoring the numerator and denominator and using the multiplicative identity.
3. Simplify $\frac{4^6}{4^3}$.
4. Simplify $\frac{17^8}{17^5}$.
5. Simplify $\frac{11^4}{11}$.
6. Simplify $\frac{3^7 \cdot 3^9}{3^4 \cdot 3^5}$.
7. What is the simplest exponential notation for $3^5 \div 3^2$? If you are not certain, write a fractional phrase for $3^5 \div 3^2$, and then simplify.
8. What is the simplest exponential notation for the quotient $5^4 \div 5^2$?
9. What is the simplest exponential notation for the quotient $4^6 \div 4^3$?
10. If $a \neq 0$, what is the simplest exponential notation for $a^6 \div a^2$?
11. What idea do you have regarding division using exponential notation?
12. Try your idea on these:
    a. $10^5 \div 10^2$
    b. $8^9 \div 8^6$
    c. $2^5 \div 2^3$
    d. $15^{25} \div 15^{20}$
    e. $7^4 \div 7^4$
    f. $8^5 \div 8^1$
    g. $2^1 \div 2^1$
    h. $4^0 \div 4^0$
13. Check your method by doing exercise 12-c by two other methods.
    a. By writing a fractional phrase for $2^5 \div 2^3$, and then simplifying.

b. By finding decimal numerals for 2^5 and 2^3, and then dividing.

14. Did you get the same results for examples 13-a and 13-b that you got for example 12-c?

Division and multiplication are opposite operations. To multiply, using exponential notation, we added exponents when the bases were the same. Since subtraction is the opposite of addition, it seems reasonable that, when the bases are the same, we should subtract exponents in order to divide. You probably discovered this for yourself in the previous exercises. This is another important fact about exponents.

For any non-zero number **a**, *and any whole numbers* **x**, **y**, *where* **x > y**,

$$a^x \div a^y = a^{x-y} \quad \text{or,}$$

$$\frac{a^x}{a^y} = a^{x-y}.$$

EXERCISES

1. Divide. Write the simplest exponential notation for each answer:

a. $\frac{3^5}{3^2}$

b. $\frac{4^8}{4^4}$

c. $\frac{5^6}{5}$

d. $\frac{8^4}{8^1}$

e. $\frac{12^3}{12^2}$

f. $\frac{7^9}{7^3}$

g. $\frac{2^6}{2^4}$

h. $\frac{4^{12}}{4^{10}}$

i. $\frac{4^5 \times 4^5}{4^2 \times 4^3}$

j. $\frac{7^3 \times 7^2}{7^4}$

k. $\frac{4^8 \times 4^5}{4^2 \times 4^3}$

l. $\frac{5^5 \times 5^3 \times 5^4}{5 \times 5^7}$

2. Factor each number completely, then write exponential notation for it.

Example:

$$500 = 5 \cdot 5 \cdot 5 \cdot 2 \cdot 2 = 5^3 \cdot 2^2$$

a. 125

b. 16

c. 225

d. 150

e. 18

f. 72

g. 400

h. 484

i. 9000

j. 1800

3. Write the simplest exponential notation for each of the following:

Examples:

$$6^2 \cdot 6^3 = 6^5$$
$$3^5 \cdot 2 \cdot 3^4 \cdot 3^0 \cdot 2^4 = 3^9 \cdot 2^5$$

a. $2^3 \cdot 2^5$
b. $3^6 \cdot 3^4$
c. $5^2 \cdot 5^4 \cdot 5$
d. $8^0 \cdot 8^3 \cdot 8$
e. $2^3 \cdot 5^2 \cdot 3^4 \cdot 2^5 \cdot 5^0 \cdot 6^0$
f. $x^3 \cdot x^2$
g. $a^2 \cdot a^7 \cdot x^5 \cdot x^3 \cdot a$
h. $\frac{3^5}{3^2}$
i. $5^8 \div 5^3$
j. $\frac{2^4}{2^0}$
k. $3^6 \div 3$
l. $\frac{5^6}{5^5}$
m. $\frac{2 \cdot 3^4 \cdot 2^5}{3^2 \cdot 2^4}$
n. $\frac{x^3}{x}, x \neq 0$
o. $c^5 \div c^3$, $c \neq 0$
p. $a^4 \cdot a^5$
q. $\frac{b^y}{b^x}$, $b \neq 0$ and $y > x$
r. $\frac{3^0 \cdot 2^5 \cdot 5^2 \cdot 3^2 \cdot 5^7}{3 \cdot 2^4 \cdot 5^9}$
s. $\frac{a^x \cdot a^y}{a^z}, a \neq 0$

4. Write expanded numerals for each of these numbers. Include zero exponents:

a. 46347
b. 32132_5
c. 65243_7
d. 101101_2

NEGATIVE INTEGERS AS EXPONENTS

We have defined positive integers and zero as exponents. How might we use negative integers as exponents?

3.3 Let's Explore

1. Copy and complete the following table:

		Find Factors of 1	Simplify
$\frac{5^3}{5^5}$	$\frac{5\times5\times5\times1}{5\times5\times5\times5\times5}$	$\frac{5}{5}\times\frac{5}{5}\times\frac{5}{5}\times\frac{1}{5\times5}$	$\frac{1}{5^2}$
$\frac{2^7}{2^8}$	$\frac{2\times2\times2\times2\times2\times2\times2\times1}{2\times2\times2\times2\times2\times2\times2\times2}$	$\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{2}{2}\times\frac{1}{2}$	$\frac{1}{2}$

$\frac{3^2}{3^5}$	$\frac{3\times3\times1}{3\times3\times3\times3\times3}$	$\frac{3}{3}\times\frac{3}{3}\times\frac{1}{3\times3\times3}$	$\frac{1}{3^3}$
$\frac{4}{4^4}$			
$\frac{10^2}{10^6}$			
$\frac{8^6}{8^{11}}$			
$\frac{a^4}{a^6}, a \neq 0$			

2. Remember how we use exponents for dividing. Suppose we try subtracting exponents on $\frac{5^3}{5^5}$. (We do not have such a rule for this case, but let's *try* it and see what happens.) We get 5^{3-5}, or 5^{-2}. Copy this table. Use this idea of subtracting and fill in the table.

	Subtract Exponents	Simplify
$\frac{5^3}{5^5}$	5^{3-5}	5^{-2}
$\frac{2^7}{2^8}$	2^{7-8}	
$\frac{3^2}{3^5}$		
$\frac{4}{4^4}$		
$\frac{10^2}{10^6}$		
$\frac{8^6}{8^{11}}$		
$\frac{a^4}{a^6}, a \neq 0$		

3. Compare the tables in exercises 1 and 2. Are the columns on the left alike? Are the columns on the right alike?

4. What conclusion can you make about how negative exponents should be defined if our rule for division using exponents is to hold?

We have seen how to define negative exponents if we want our rule for division to hold. Let's define them that way.

> *For any non-zero number of arithmetic* **a** *and any positive integer* **x**,
>
> ‘a^{-x}’ *means* ‘$\frac{1}{a^x}$’
>
> *(the multiplicative inverse, or reciprocal, of* a^x*).*

EXERCISES

1. Write exponential notation for each of these numbers:

Example:

$$\frac{1}{4^2} = 4^{-2}$$

a. $\frac{1}{5^4}$
b. $\frac{1}{4^7}$
c. $\frac{1}{10^3}$
d. $\frac{1}{12^4}$
e. $\frac{1}{6^7}$
f. $\frac{1}{8^5}$
g. $\frac{1}{9^5}$
h. $\frac{1}{4^5}$

2. Write fractional numerals for each of these numbers:

Example:

$$3^{-5} = \frac{1}{3^5}$$

a. 4^{-2}
b. 8^{-5}
c. 3^{-12}
d. 5^{-5}
e. 10^{-3}
f. 7^{-8}
g. 8^{-3}
h. 11^{-7}

In our definition of negative exponents we said that ‘x’ represents a positive integer. This is so that ‘$-x$’ will represent a negative integer. Then we say that $a^{-x} = \frac{1}{a^x}$, for any non-zero number a. Suppose ‘x’ represents a negative integer, for instance -3. Would the sentence still be true? We would have

$$a^{-(-3)} = \frac{1}{a^{-3}}, \text{ or } a^3 = \frac{1}{a^{-3}}.$$

If this sentence is true it would suggest that whenever we replace an exponent in a^x by its additive inverse, we get the reciprocal of a^x.

3.4 Let's Explore

1. Consider $\frac{1}{3^{-2}}$.
   - a. What is the meaning of '3^{-2}', by definition?
   - b. In '$\frac{1}{3^{-2}}$', replace '3^{-2}' by your answer to a.
   - c. You should now have a fractional numeral with numerator 1 and denominator $\frac{1}{3^2}$. Multiply by $\frac{3^2}{3^2}$ and simplify.
2. Consider $\frac{1}{5^{-3}}$.
   - a. What is the meaning of '5^{-3}', by definition?
   - b. In '$\frac{1}{5^{-3}}$', replace '5^{-3}' by your answer to a.
   - c. Multiply by $\frac{5^3}{5^3}$ and simplify.
3. Consider $\frac{1}{4^{-3}}$. Simplify as in exercises 1 and 2.
4. Consider $\frac{1}{a^{-x}}$, where $a \neq 0$ and x is a positive integer.
   - a. What is the meaning of 'a^{-x}', by definition?
   - b. In '$\frac{1}{a^{-x}}$', replace 'a^{-x}' by your answer to a.
   - c. Multiply by $\frac{a^x}{a^x}$ and simplify.

You have now seen that $\frac{1}{a^{-x}} = a^x$ for any positive integer x and for any non-zero number a. This means that when we change a *positive* exponent in a^x to its additive inverse we get the reciprocal of a^x. By our definition we know that when we replace a *negative* exponent by its additive inverse we get the reciprocal of a^x.

Suppose an exponent is zero. Since $a^0 = 1$ for all non-zero a, the reciprocal of a^0 is a^0. If we replace a *zero* exponent by its additive inverse, we still get the reciprocal of a^0.

We now have proved a general rule for integers as exponents:

For any non-zero number of arithmetic **a**, *and* **any** *integer* **x**,

$$a^{-x} = \frac{1}{a^{x}}, \text{ and also } a^{x} = \frac{1}{a^{-x}}.$$

(Replacing the exponent by its additive inverse gives a reciprocal.)

EXERCISES

1. Write exponential notation for each of these numbers.

Example:

$$\frac{1}{4^{-2}} = 4^{2}.$$

a. $\frac{1}{5^{-3}}$
b. $\frac{1}{6^{2}}$
c. $\frac{1}{3^{0}}$
d. $\frac{1}{9^{5}}$
e. $\frac{1}{17^{3}}$
f. $\frac{1}{10^{-5}}$
g. $\frac{1}{7^{0}}$
h. $\frac{1}{8^{-21}}$

2. Write fractional numerals for each of these numbers.

Example:

$$3^{5} = \frac{1}{3^{-5}}$$

a. 4^{2}
b. 8^{-5}
c. 7^{0}
d. 9^{-4}
e. 5^{6}
f. 10^{-4}
g. 12^{5}
h. 11^{-7}

We made a definition of negative exponents. We made it in a certain way because we wanted to be able to use them like the positive ones in calculating. We would like to be able to add exponents when multiplying and subtract exponents when dividing. Do we now have these properties? We do not yet know. We know that *if* the properties are to hold, *then* we must make the definition as we did. The converse of this is: *If* we make the definition as we did, *then* the properties hold. The converse might be false.

Let's look further, and see if we have the properties we would like.

3.5 Let's Explore

1. Copy and complete this table. Look for patterns. Extend the table with examples of your own if necessary.

	Definition of negative exponent	Multiply and simplify	Go back to first column. Add exponents	Simplify
$3^3 \cdot 3^{-2}$	$3^3 \cdot \frac{1}{3^2}$	3^1	3^{3+-2}	3^1
$4^5 \cdot 4^{-2}$				
$5^7 \cdot 5^{-3}$				
$2^5 \cdot 2^{-3}$				

2. Copy and complete this table. Look for patterns. Extend the table with examples of your own if necessary.

	Definition of negative exponent	Multiply and simplify	Write exponential notation	Go back to first column. Add exponents	Simplify
$3^{-3} \cdot 3^2$	$\frac{1}{3^3} \cdot 3^2$	$\frac{1}{3^1}$	3^{-1}	3^{-3+2}	3^{-1}
$4^{-5} \cdot 4^2$					
$5^{-7} \cdot 5^3$					
$2^{-5} \cdot 2^3$					
$3^{-4} \cdot 3^0$					

3. Copy and complete this table. Look for patterns. Extend the table with examples of your own if necessary.

	Definition of negative exponents	Multiply and simplify	Write exponential notation	Go back to first column. Add exponents	Simplify
$3^{-2} \cdot 3^{-3}$	$\frac{1}{3^2} \cdot \frac{1}{3^3}$	$\frac{1}{3^5}$	3^{-5}	3^{-2+-3}	3^{-5}
$4^{-5} \cdot 4^{-2}$					
$5^{-7} \cdot 5^{-3}$					
$2^{-5} \cdot 2^{-3}$					

4. From the tables on the preceding page, does it seem that the rule for multiplying using exponents works when some or all of the exponents are negative?

You have seen that the rule for multiplying using exponents seems to work when some or all exponents are negative. This is true, provided the bases are the same. Some of you may wish to prove it.

For any non-zero number ***a****, and for* ***any*** *integers* ***m*** *and* ***n****,*

$$a^m \cdot a^n = a^{m+n}.$$

EXERCISES

1. Write simplest exponential notation for each of these:
   - a. $3^2 \cdot 3^{-1}$
   - b. $5^{-2} \cdot 5^{-3}$
   - c. $3^4 \cdot 3^{-2}$
   - d. $6^4 \cdot 5^{-3} \cdot 5^2$
   - e. $7^{-2} \cdot 5^4 \cdot 5^{-6}$
   - f. $4^2 \cdot 3^{-4} \cdot 4^1$
   - g. $2^5 \cdot 5^{-2} \cdot 2^2$
   - h. $2^5 \cdot 2^4 \cdot 2^{-4}$
   - i. $2^4 \cdot 3^5 \cdot 2^{-2} \cdot 3^{-6}$
   - j. $9^4 \cdot 7^{-5} \cdot 9^3 \cdot 9^2$
   - k. $6^{-3} \cdot 5^{-2} \cdot 6^7$
   - l. $4^{-5} \cdot 3^5 \cdot 4^{-2} \cdot 3^{-3}$
   - m. $10^3 \cdot 5^{-3} \cdot 7^{-5} \cdot 10^2$
   - n. $3^4 \cdot 5^2 \cdot 3^{-7} \cdot 5^{-1}$

2. Write fractional phrases without negative exponents for each of the following:
   - a. $3^5 \cdot 2^{-3}$
   - b. $4 \cdot a^3 \cdot x^{-2} \cdot y^5$
   - c. $5^{-3} \cdot a^5 \cdot b^{-2}$
   - d. $2^5 \cdot 3^{-4} \cdot 6^3$
   - e. $3^{-2} \cdot 5^{-6} \cdot s^{-4} \cdot 6^{-1}$
   - f. $(4^5 \cdot x^{-3}) \cdot (4^{-2} \cdot x^5)$
   - g. $(3^{-7} \cdot y^2) \cdot (3^9 \cdot y^{-1})$
   - h. $(7^5 \cdot x^3 \cdot y^6) \cdot (7^{-5} \cdot x^{-2} \cdot y^{-10})$
   - i. $(4^{-5} \cdot x^{-6} \cdot y^{-7}) \cdot (4^{-2} \cdot x^6 \cdot y^9 \cdot z^2)$
   - j. $(3^{-5} \cdot x^{-5} \cdot y^{-10}) \cdot (3^7 \cdot x^{-2} \cdot y^8 \cdot z^3)$

We have discovered that the rule of adding exponents works in multiplying, but we have not yet seen whether the rule of subtraction of exponents works in dividing.

3.6 Let's Explore

1. Copy and complete the table at the top of the next page. Look for patterns. Extend the table with examples of your own if necessary.

	Definition of division	Definition of negative exponent	Add exponents	Go back to first column. Subtract exponents	Simplify
$\frac{3^2}{3^{-4}}$	$3^2 \cdot \frac{1}{3^{-4}}$	$3^2 \cdot 3^4$	3^6	$3^{2\,-\,-4}$	3^6
$\frac{5^3}{5^{-7}}$					
$\frac{4^2}{4^{-1}}$					
$\frac{3^7}{3^{-7}}$					
$\frac{7^5}{7^{-4}}$					

2. Copy and complete this table. Look for patterns. Extend the table if necessary.

	Definition of division	Definition of negative exponents	Simplify	Write exponential notation	Go back to first column. Subtract exponents	Simplify
$\frac{3^{-2}}{3^3}$	$3^{-2} \cdot \frac{1}{3^3}$	$\frac{1}{3^2} \cdot \frac{1}{3^3}$	$\frac{1}{3^5}$	3^{-5}	$3^{-2\,-\,3}$	3^{-5}
$\frac{4^{-1}}{4^2}$						
$\frac{5^{-5}}{5^2}$						
$\frac{7^{-1}}{7^3}$						
$\frac{2^{-6}}{2^6}$						

3. Copy and complete the table at the top of the next page. Look for patterns. Extend the table with examples of your own if necessary.

	Definition of negative exponents	Definition of division (multiply by reciprocal)	Simplify	Go back to first column. Subtract exponents	Simplify
$\frac{3^{-2}}{3^{-3}}$	$\dfrac{\frac{1}{3^2}}{\frac{1}{3^3}}$	$\frac{1}{3^2} \cdot 3^3$	3^1	$3^{-2\,-\,-3}$	3^1
$\frac{4^{-3}}{4^{-5}}$					
$\frac{5^{-4}}{5^{-2}}$					
$\frac{7^{-6}}{7^{-2}}$					
$\frac{2^{-4}}{2^{-4}}$					

4. Copy and complete this table. Look for patterns. Extend the table if necessary.

	Definition of negative and zero exponents	Simplify	Write exponential notation	Go back to first column. Subtract exponents	Simplify
$\frac{4^{-3}}{4^0}$	$\dfrac{\frac{1}{4^3}}{1}$	$\frac{1}{4^3}$	4^{-3}	$4^{-3\,-\,0}$	4^{-3}
$\frac{3^{-2}}{3^0}$					
$\frac{5^0}{5^{-3}}$					
$\frac{7^0}{7^{-5}}$					

5. From these tables, does it seem that the rule for dividing works when one or both exponents are negative?

We have seen that the rule for dividing using exponents probably holds when one or both exponents are negative. It actually does. (Of course the bases must be alike.) You may wish to try proving this.

For any non-zero number **a**, *and for* **any** *integers* **m** *and* **n**,

$$\frac{a^m}{a^n} = a^{m-n}.$$

Have you wondered why we do not define '0^0'? Look at the above statement, and suppose that $a = 0$, $m = 3$ and $n = 3$. Then we would have

$$\frac{0^3}{0^3} = 0^{3-3}, \text{ or } 0^0.$$

But $0^3 = 0$, so this would mean that

$$0^0 = \frac{0}{0},$$

and we know that we cannot divide by 0. To avoid this difficulty we agree to leave '0^0' undefined, or meaningless.

EXERCISES

1. Write exponential notation for each of these:

Example:

$$\frac{2}{a^3b} = 2 \cdot a^{-3}b^{-1}$$

a. $\frac{1}{a^3 \cdot b^2}$

b. $\frac{2}{x^2 \cdot y^2}$

c. $\frac{3^2}{2^4 \cdot 5^2}$

d. $\frac{s^3 \cdot t}{u^2}$

e. $\frac{10^2 \cdot a}{b}$

f. $\frac{x}{3^{-4} \cdot b}$

2. Simplify the following. Write two different phrases for each, (1) a fractional phrase and (2) a phrase with one or more negative exponents.

Example 1:

$$3^4 \cdot 2^{-1} \cdot 3^2 = \frac{3^{4+2}}{2^1} = \frac{3^6}{2} \text{ (one answer)}$$

$$3^4 \cdot 2^{-1} \cdot 3^2 = 3^{4+2} \cdot 2^{-1} = 3^6 \cdot 2^{-1} \text{ (second answer)}$$

Example 2:

$$\frac{7^2 \cdot a^4}{7^5 \cdot a^2} = 7^2 \cdot 7^{-5} \cdot a^4 \cdot a^{-2}$$

$$= 7^{-3} \cdot a^2 \text{ (one answer)}$$

$$= \frac{a^2}{7^3} \qquad \text{(second answer)}$$

a. $2^3 \cdot 2^{-6} \cdot 5^3$

b. $3^2 \cdot 5^3 \cdot 3^{-5} \cdot 5^2$

c. $7^{-4} \cdot 9^5 \cdot 7^{-6} \cdot 9^{-2}$

d. $4^5 \cdot 4^{-2} \cdot 7^{-3} \cdot 7^2$

e. $\frac{9^3 \cdot b^5}{9^2 \cdot b^8}$

f. $\frac{5^2 \cdot a^5}{5^7 \cdot a^2}$

g. $\frac{3^5 \cdot a^2 \cdot b^7}{3^2 \cdot a^4 \cdot b^5}$

h. $\frac{7^{10} \cdot x^3 \cdot y^9}{7^5 \cdot x^8 \cdot y^4}$

i. $\frac{10^3 \cdot a^5 \cdot b^7}{10^5 \cdot a^2 \cdot b^9}$

j. $\frac{8^2 \cdot s^5 \cdot t^8}{8^6 \cdot s^3 \cdot t^{10}}$

k. $\frac{3^2 \cdot 2^5 \cdot a^4 \cdot b \cdot 3^0}{2^6 \cdot b^3 \cdot a^0}$

l. $\frac{10^2 \cdot 10^3 \cdot 10^5}{10^7 \cdot 10^3 \cdot 10^6}$

m. $\frac{x^3 \cdot y^2 \cdot s}{s^3 \cdot x \cdot y^5 \cdot 3}$

n. $\frac{x^4 \cdot y^{-2} \cdot s}{x^{-1} \cdot y^2 \cdot z^3}$

EXPANDED NUMERALS

We defined the zero exponent to fit a pattern for expanded numerals. We then found that it worked for multiplying or dividing with exponential notation. We defined negative exponents so that we could multiply and divide easily. How will negative exponents work for expanded numerals? In the decimal numeral '8724.563' the '5' represents a number of tenths, the '6' a number of hundredths, and the '3' a number of thousandths.

Keeping the usual pattern of decreasing exponents, let's write a symbol like an expanded numeral:

$$8 \times 10^3 + 7 \times 10^2 + 2 \times 10^1 + 4 \times 10^0 + 5 \times 10^{-1} + 6 \times 10^{-2} + 3 \times 10^{-3}.$$

Here '5×10^{-1}' means $5 \times \frac{1}{10}$, or .5.

'6×10^{-2}' means $6 \times \frac{1}{10^2}$, or $6 \times \frac{1}{100}$, or .06.

'3×10^{-3}' means $3 \times \frac{1}{10^3}$, or $3 \times \frac{1}{1000}$, or .003.

We can see how the symbol we wrote really is an expanded numeral for 8724.563. Actually our definition of negative exponents works for all expanded numerals.

EXERCISES

Write expanded numerals for these numbers:

1. 3562.143
2. 89724.10342
3. 35.10023
4. 105.00352
5. .02145
6. 324.3421_5
7. 6453.21046_7
8. 1011.101101_2

A POWER OF A POWER

The third power of 5 is 125. It may be named '5^3'. The fifth power of the third power of 5 may be named '125^5', or '$(5^3)^5$'. We say that $(5^3)^5$ is *the fifth power of the third power of 5.*

3.7 Let's Explore

1. Copy and complete this table:

4^3	$4 \times 4 \times 4$
25^4	$25 \times 25 \times 25 \times 25$
49^2	
121^6	
9^8	

2. Write the prime factorization of each number. Copy and complete the table.

4	2^2
25	
49	
121	
9	

3. Since $4^3 = 4 \times 4 \times 4$, and $4 = 2^2$, we know that $4^3 = 2^2 \times 2^2 \times 2^2$. In a similar way, write a name for each of the following. Fill in the blanks.
$25^4 = \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$
$49^2 = \_\_\_ \times \_\_\_$
$121^6 = \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$
$9^8 = \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_ \times \_\_\_$

4. Write the simplest exponential notation for each of the numbers in exercise 3. Copy and complete the table.

4^3	$(2^2)^3$	$(2^2) \times (2^2) \times (2^2) = 2^{2+2+2}$	2^6
25^4			
49^2			
121^6			
9^8			

5. Look closely at the list of numerals you have written in exercise 4. What would be an easy way to simplify $(2^2)^3$? Try your idea on each of the other examples. Does it work? Did you get the same results as in exercise 4?

6. Use your idea to simplify the following:

a. $(10^3)^4$	f. $(a^3)^5$
b. $(5^4)^3$	g. $(5^0)^3$
c. $(2^2)^{10}$	h. $(4^1)^3$
d. $(3^4)^4$	i. $(7^2)^0$
e. $(7^6)^3$	j. $(6^2)^1$

You should have discovered that sometimes we can raise a power to a power by multiplying the exponents. For example:

$$(5^2)^3 = 5^2 \cdot 5^2 \cdot 5^2 = 5^{2+2+2} = 5^{3 \times 2} = 5^6$$

$$(2^3)^5 = 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 \cdot 2^3 = 2^{3+3+3+3+3} = 2^{5 \times 3} = 2^{15}$$

By definition of the zero exponent, $(5^3)^0 = 1$. If we multiply our exponents this time, we get $(5^3)^0 = 5^{3 \times 0} = 5^0 = 1$. Similarly,

$$(4^0)^3 = 1^3 = 1, \text{ and}$$

$$(4^0)^3 = 4^{0 \times 3} = 4^0 = 1.$$

For any non-zero arithmetic number ***a****, and any whole numbers* ***m*** *and* ***n****,*

$$(a^m)^n = a^{m \cdot n}.$$

EXERCISES

Simplify:

1. $(3^5)^3$
2. $(4^7)^6$
3. $(12^5)^3$
4. $(7^0)^4$
5. $(10^4)^0$
6. $(127^0)^0$
7. $(2^8)^3$
8. $(11^{14})^3$
9. $(17^4)^{21}$
10. $(x^4)^5$
11. $(t^0)^4$
12. $(s^6)^3$

So far in raising a power to a power, we have used only exponents which are positive integers or zero. We found we could multiply exponents in these cases. We have not studied multiplication for negative integers. Perhaps we can find an easy way to raise a power to a power when one or both of the exponents is negative.

3.8 Let's Explore

1. Using exponential notation, write fractional phrases for each of the following. Copy and complete the table:

3^{-2}	$\frac{1}{3^2}$
2^{-4}	
3^{-5}	
7^{-3}	
5^{-4}	

2. Copy and complete this table:

$(3^{-2})^3$	$\frac{1}{3^2} \times \frac{1}{3^2} \times \frac{1}{3^2}$	$\frac{1}{3^6}$	3^{-6}
$(2^{-4})^2$	$\frac{1}{2^4} \times \frac{1}{2^4}$	$\frac{1}{2^8}$	2^{-8}
$(3^{-5})^5$			
$(7^{-3})^4$			
$(5^{-4})^6$			

3. Do you see a quick way to simplify a phrase like $(7^{-2})^5$? Use your idea on these:

a. $(10^{-3})^2$
b. $(4^{-6})^3$
c. $(7^{-8})^4$
d. $(4^{-2})^5$
e. $(a^{-7})^3$
f. $(p^{-8})^4$

By now your have seen how to simplify expressions (phrases) like $(7^{-2})^5$. By definition of negative exponents, ‘$(7^{-2})^5$’ means the same as ‘$\left(\frac{1}{7^2}\right)^5$’. Suppose the other exponent were the negative one. What would the symbol mean? By definition of negative exponents, ‘$(7^2)^{-5}$’ means the same as ‘$\frac{1}{(7^2)^5}$.’ How do these compare?

3.9 Let's Explore

1. Remember that $(2^6)^{-3} = \frac{1}{(2^6)^3}$. Copy and complete this table:

$(3^2)^{-3}$	$\frac{1}{(3^2)^3}$	$\frac{1}{3^6}$	3^{-6}
$(2^4)^{-2}$			
$(3^5)^{-5}$			
$(7^3)^{-4}$			
$(5^4)^{-6}$			

2. Do you see a quick way to simplify expressions like $(2^6)^{-3}$? Use your idea on these.

a. $(7^3)^{-2}$
b. $(5^4)^{-6}$
c. $(8^4)^{-5}$
d. $(3^5)^{-3}$
e. $(4^4)^{-4}$
f. $(2^5)^{-6}$
g. $(5^0)^{-3}$
h. $(3^{-2})^0$

You have probably discovered a quick way to find a power of a power, when one exponent is negative. We can say to multiply the absolute values of the exponents and then take the additive inverse. For example,

$$(3^{-2})^4 = 3^{-8}.$$

We multiply $|{}^{-}2|$ by $|4|$ and get 8. Then we take the additive inverse and get ${}^{-}8$. Also, to find $(3^2)^{-4}$ we multiply $|2|$ by $|{}^{-}4|$ and get 8. Then we take the additive inverse and get ${}^{-}8$.

For any non-zero number of arithmetic ***a****, and for any whole numbers* ***m*** *and* ***n****,*

$$(a^{-m})^n = a^{-m \cdot n} \text{ and } (a^m)^{-n} = a^{-m \cdot n}.$$

EXERCISES

Simplify:

1. $(3^{-4})^3$
2. $(4^{-5})^2$
3. $(5^3)^{-4}$
4. $(2^{-3})^6$
5. $(7^4)^{-7}$
6. $(3^2)^{-5}$
7. $(6^{-7})^6$
8. $(p^3)^{-6}$
9. $(x^{-5})^7$
10. $(4^2)^{-5}$
11. $(6^5)^{-2}$
12. $(11^4)^{-3}$

Review Practice

Multiply. Don't forget to estimate:

1. $\begin{array}{r} 315 \\ \underline{24} \end{array}$
2. $\begin{array}{r} 712 \\ \underline{41} \end{array}$
3. $\begin{array}{r} 318 \\ \underline{27} \end{array}$
4. $\begin{array}{r} 47.6 \\ \underline{2.15} \end{array}$
5. $\begin{array}{r} 6.15 \\ \underline{1.24} \end{array}$
6. $\begin{array}{r} 47.3 \\ \underline{6.51} \end{array}$
7. $\begin{array}{r} 38.12 \\ \underline{8.04} \end{array}$
8. $\begin{array}{r} 6.713 \\ \underline{3.11} \end{array}$
9. $\begin{array}{r} 751.2 \\ \underline{4.33} \end{array}$
10. $\begin{array}{r} 357.8 \\ \underline{5.23} \end{array}$
11. $\begin{array}{r} .3105 \\ \underline{1.44} \end{array}$
12. $\begin{array}{r} .01246 \\ \underline{.312} \end{array}$

We have not yet studied a case like $(2^{-3})^{-4}$, where both exponents are negative.

3.10 Let's Explore

1. Copy and complete this table:

	Definition of negative exponents	Simplify	Definition of negative exponents
$(5^{-2})^{-3}$	$\frac{1}{(5^{-2})^{3}}$	$\frac{1}{5^{-6}}$	5^{6}
$(2^{-3})^{-4}$			
$(3^{-2})^{-5}$			
$(2^{-4})^{-3}$			
$(3^{-5})^{-3}$			
$(5^{-6})^{-3}$			
$(5^{-3})^{-8}$			
$(7^{-2})^{-4}$			

2. Have you now discovered a way to simplify a phrase like '$(7^{-3})^{-5}$'? Explain it.

You have probably discovered a quick way to simplify phrases like '$(7^{-3})^{-5}$'. To simplify phrases like '$(7^{-3})^{-5}$', multiply the absolute values of the exponents. If we multiply $|{}^{-}3|$ by $|{}^{-}5|$ we get 15, the exponent of the equivalent but simpler phrase '7^{15}'. The new exponent in these cases is always positive.

For any non-zero number of arithmetic ***a****, and for any whole numbers* ***m*** *and* ***n****,*

$$(a^{-m})^{-n} = a^{m \cdot n}.$$

We know that by definition of the exponent ${}^{-}n$,

$$\text{'}(a^{-m})^{-n}\text{' means the same as } \text{'}\frac{1}{(a^{-m})^{n}}\text{'}.$$

Also, the denominator,

$$\text{'}(a^{-m})^{n}\text{', means the same as '}a^{-m \cdot n}\text{'}.$$

Therefore

$$\frac{1}{(a^{-m})^{n}} = \frac{1}{a^{-m \cdot n}}.$$

But by definition of negative exponents,

$$\frac{1}{a^{-m \cdot n}} = a^{m \cdot n}.$$

Therefore

$$(a^{-m})^{-n} = a^{m \cdot n}.$$

This discussion is one way of proving the statement on the preceding page. Did you give proofs of earlier statements? Perhaps you should do so now.

EXERCISES

1. Simplify:
   - a. $(4^{-3})^{-5}$
   - b. $(6^{-2})^{-1}$
   - c. $(3^{-4})^{-2}$
   - d. $(8^{-1})^{-5}$
   - e. $(x^{-4})^{-6}$
   - f. $(t^{-3})^{-2}$
   - g. $(3^{-1})^{-4} \times (4^{-2})^{-5}$
   - h. $(x^{-2})^{-3} \times (y^{-5})^{-2}$

2. Prove, using definitions and properties of exponents, that:
   - a. $(3^{-2})^{4} = 3^{-8}$
   - b. $(5^{-6})^{3} = 5^{-18}$
   - c. $(5^{3})^{-2} = 5^{-6}$
   - d. $(4^{7})^{-3} = 4^{-21}$

*3. a. Prove that $(a^{-m})^{n} = a^{-mn}$, for any non-zero number a, and for any natural numbers m and n.

b. Prove that $(a^{m})^{-n} = a^{-mn}$, for any non-zero number a, and for any natural numbers m and n.

MULTIPLICATION OF INTEGERS

When we studied integers, we did not define multiplication. We defined addition and subtraction only. We do know how to multiply integers if they are not negative. How might we define multiplication for negative integers and for zero? Finding powers of powers has shown us one possible way.

Let's use the ideas we have just studied, and define multiplication of integers accordingly.

We have already defined multiplication for positive integers (natural numbers). For a positive and a negative integer we make the following definition.

*For any natural numbers **a** and **b**,*
$$^{-}a \cdot b = {}^{-}(a \cdot b).$$

For two negative integers we make the definition as follows:

*For any natural numbers **a** and **b**,*
$$^{-}a \cdot {}^{-}b = a \cdot b.$$

The product of 0 and *any* integer (positive, negative or zero) is 0. We say this as follows:

*For any integer **a**,*
$$a \cdot 0 = 0.$$

To find the product of a positive integer and a negative integer, we multiply their absolute values and then take the additive inverse. The product is negative. To find the product of two negative integers, we simply multiply their absolute values. The product is positive.

EXERCISES

Multiply:

1. $^{-}2 \cdot 3$
2. $5 \cdot {}^{-}6$
3. $4 \cdot {}^{-}5$
4. $^{-}3 \cdot {}^{-}4$
5. $^{-}7 \cdot {}^{-}5$
6. $^{-}9 \cdot {}^{-}4$
7. $3 \cdot ({}^{-}5 \cdot {}^{-}2)$
8. $^{-}2 \cdot (4 \cdot {}^{-}3)$
9. $^{-}7 \cdot ({}^{-}2 \cdot 3)$
10. $({}^{-}4 \cdot 6) \cdot ({}^{-}5 \cdot {}^{-}3)$
11. $({}^{-}5 \cdot 3) \cdot (4 \cdot {}^{-}7)$
12. $({}^{-}1 \cdot {}^{-}5) \cdot ({}^{-}3 \cdot {}^{-}4)$
13. $^{-}7 \cdot 0 \cdot 2$
14. $^{-}4 \cdot 0 \cdot {}^{-}3$
15. $4 \cdot 5 \cdot 0$

Now that we have defined multiplication for integers, we can more simply state a rule for powers of powers.

> *For any non-zero number of arithmetic* ***a****, and for* ***any integers x*** *and* ***y****,*
>
> $$(a^x)^y = a^{x \cdot y}.$$

EXERCISES

Write simplest exponential notation for these numbers:

1. $(5^2)^3$
2. $(5^3)^2$
3. $(7^{-4})^3$
4. $(7^4)^3$
5. $(11^{-5})^{-2}$
6. $(11^5)^{-2}$
7. $(11^{-5})^2$
8. $(11^5)^2$
9. $(2^3)^5$
10. $(2^{-3})^{-5}$
11. $(x^{-2})^{-4}$
12. $(s^{-8})^5$
13. $(3^2)^5 \cdot 3^4$
14. $(2^4)^2 \cdot (2^3)^3$
15. $(10^3)^5 \cdot (10^2)^3 \cdot 10^5$
16. $(2^{-3})^2 \cdot (3^5)^3 \cdot (2^{-4})^{-5}$
17. $(3^{-2})^0 \cdot (4^2)^{-3} \cdot (5^0)^{-5}$
18. $\dfrac{(10^2)^3 \cdot (10^4)^3}{(10^5)^2}$
19. $\dfrac{(3^4)^5 \cdot (2^4)^4 \cdot (5^2)^3}{(5^3)^2 \cdot (2^4)^5 \cdot (3^5)^4}$
20. $\dfrac{(a^5)^6 \cdot (b^4)^2 \cdot a^7 \cdot b^9}{(a^5)^8 \cdot (b^5)^2}$
    (assume $a \neq 0$ and $b \neq 0$)
21. $\dfrac{(x^3)^2 \cdot (y^5)^2 \cdot x^8}{(a^3)^5 \cdot y^4}$
    (assume $a \neq 0$ and $y \neq 0$)
22. $(a^5)^3 \cdot (b^{-2})^6 \cdot (a^3)^{-4}$
    (assume $a \neq 0$ and $b \neq 0$)
23. $(x^4)^{-2} \cdot (x^{-3})^{-4} \cdot (x^{-2})^4$
    (assume $x \neq 0$)

*24. $(s^3)^7 \cdot (t^{-5})^{-2} \cdot (s^7)^{-3}$
(assume $s \neq 0$ and $t \neq 0$)

*25. $\dfrac{(2^4)^4 \cdot (3^5)^2 \cdot 5^4}{(3^{-2})^{-5} \cdot (2^{-2})^{-8}}$

26. $(a^3)^{-2} \cdot (b^{-3})^{-6} \cdot (a^{-6})^{-3} \cdot (b^3)^{-7}$
(assume $a \neq 0$ and $b \neq 0$)

COMMUTATIVE PROPERTY FOR MULTIPLICATION OF INTEGERS (OPTIONAL)

We know that we have the commutative property for multiplication of whole numbers. Do we have it for all integers?

3.11 Let's Explore

1. We know that $a \cdot b = b \cdot a$, where a and b are any whole numbers. What do we call this property? Do you think it holds for integers?
2. Suppose a and b are any natural numbers. What is another phrase for ${}^{+}a \cdot {}^{-}b$? What is another phrase for ${}^{-}b \cdot {}^{+}a$? Did you get two names for the same number? Explain.
3. Suppose a and b are any natural numbers. What is another phrase for ${}^{-}a \cdot {}^{+}b$? What is another phrase for ${}^{+}b \cdot {}^{-}a$? Do the phrases name the same number? Explain.
4. Suppose a and b are any natural numbers. What is another phrase for ${}^{-}a \cdot {}^{-}b$? What is another phrase for ${}^{-}b \cdot {}^{-}a$? Do the phrases name the same number? Explain.

Since we have tried *all* of the different cases, we can say that the commutative property holds for multiplication of integers. Some of you may wish to determine whether or not we have closure, an identity, inverses, and associativity for multiplication of the integers. Do not say yes too quickly. You may be surprised at what you find out. Try to find proofs.

MULTIPLYING AND DIVIDING BY POWERS OF 10

Powers of ten are important in decimal numeration. Let's think about powers of ten with exponential notation and see what patterns we can find.

3.12 Let's Explore

1. Find the following products:

$$\begin{array}{r} 32 \\ \underline{\times 10} \end{array} \qquad \begin{array}{r} 3.2 \\ \underline{\times 10} \end{array} \qquad \begin{array}{r} .032 \\ \underline{\times 10} \end{array} \qquad \begin{array}{r} 3200 \\ \underline{\times 10} \end{array} \qquad \begin{array}{r} .00032 \\ \underline{\times 10} \end{array}$$

2. Find the following products:

$$\begin{array}{r} 563 \\ \underline{\times 100} \end{array} \qquad \begin{array}{r} 56.3 \\ \underline{\times 100} \end{array} \qquad \begin{array}{r} .563 \\ \underline{\times 100} \end{array} \qquad \begin{array}{r} .0563 \\ \underline{\times 100} \end{array} \qquad \begin{array}{r} 5.63 \\ \underline{\times 100} \end{array}$$

3. Remember that $100 = 10 \cdot 10$ and that $10 \cdot 10 = 10^2$.
Find the following products:

$$\begin{array}{r} 563 \\ \underline{\times 10^2} \end{array} \qquad \begin{array}{r} 56.3 \\ \underline{\times 10^2} \end{array} \qquad .563 \times 10^2 \qquad .0563 \times 10^2 \qquad 5.63 \times 10^2$$

4. Find these products:
   246×10^4 24.6×10^3 $.246 \times 10^2$ $.246 \times 10^5$
5. Did you find a pattern? How would you use it to multiply 3.2467 by 10^3?

It would seem that we could just move the decimal point to the right the number of places given by the exception.

3.13 Let's Explore

1. Consider the numeral '$500 \cdot 10^{-2}$'.
   If you moved the decimal point in the numeral '500.0' two places to the right what numeral would you have?
2. But '$500 \cdot 10^{-2}$' is another numeral for $500 \times \frac{1}{10^2}$, which may also be named $\frac{\text{'500'}}{10^2}$. Factor the numerator and denominator of $\frac{\text{'500'}}{10^2}$; then simplify.
3. How do your answers for exercises 1 and 2 compare?
4. In what direction could you have moved the decimal point in exercise 1 to name the number in exercise 2?
5. Write a fractional numeral for $23{,}000 \cdot 10^{-3}$. Factor the numerator and denominator, and then simplify.
6. Could you have done this a shorter way? How?

This short cut for multiplying by powers of ten is very convenient to use. If the exponent of ten is positive, the decimal point moves to the right the number of places given by the exponent.

Examples:

$$3.426 \times 10^2 = 342.6$$
$$2.4837 \times 10^4 = 24{,}837$$

If, however, the exponent is negative, the decimal point moves to the left when we find a simpler numeral.

Examples:

$$3.24 \times 10^{-2} = .0324$$
$$668.56 \times 10^{-3} = .66856$$

We could also say that when we multiply by a power of ten, where the exponent is positive, the decimal point moves to the right. When we divide by a power of ten (and the exponent is positive), the decimal point moves to the left.

What happens when we divide by a power of ten, where the exponent is negative? Here is an example:

$$\frac{346.213}{10^{-2}} = 346.213 \times 10^{2} = 34621.3$$

Dividing by 10^{-2} is the same as multiplying by 10^{2}.

EXERCISES

Simplify:

Examples:

$$2.463 \times 10^{2} = 246.3$$
$$423.689 \times 10^{-2} = 4.23689$$

1. 3.468×10^{2}
2. 24.65×10^{2}
3. $.0345 \times 10^{3}$
4. 1.00056×10^{4}
5. 326.5×10^{3}
6. 327×10^{2}
7. 562.3×10^{-4}
8. $.243 \times 10^{-2}$
9. 267×10^{-3}
10. 342.78×10^{-1}
11. $\frac{4.68}{10}$
12. $\frac{4.68}{10^{-1}}$
13. $\frac{32.51}{10^{2}}$
14. $\frac{32.51}{10^{-2}}$
15. $\frac{.2634}{10^{3}}$
16. $\frac{.2634}{10^{-3}}$
17. $\frac{268.14}{10^{2}}$
18. $\frac{268.14}{10^{-2}}$
19. $\frac{.0042}{10^{3}}$
20. $\frac{.0042}{10^{-3}}$
21. 24.63×10^{3}
22. 89.2×10^{5}
23. $\frac{4563}{10^{2}}$
24. $.346 \times 10^{-6}$
25. $\frac{8.274}{10^{5}}$

26. 72.38×10^8

27. 34×10^{-10}

28. $\frac{1.23}{10^{18}}$

29. $8.4 \cdot 10^{20}$

30. $6.342 \cdot 10^{15}$

SCIENTIFIC NOTATION

In the last few exercises some numbers were so large that their numerals were difficult to read and write. Scientists frequently work with very large and very small numbers.

Astronomers, for example, work with large numbers for distances. If they used ordinary decimal numerals it would take too much time to write them. The light from the North Star must travel for more than four hundred years, at a speed of 186,284 miles per second, before it reaches the earth. Traveling at that speed, light travels about six trillion miles per year. Multiplying six trillion by 400 we find that the distance from the earth to the North Star is about *2,400,000,000,000,000* miles.

British Information Services

The Jodrell Bank radio telescope, Cheshire, England can be rotated and tilted in any direction to collect radio echoes from stars, meteors and planets at distances of up to about 1000 million light years.

Alpha Centauri, the closest star to the earth, is about *25,800,000,000,000* miles away.

The galaxy nearest our own galaxy is the Great Nebula in the Constellation of Andromeda. The distance from the earth to the Great Nebula is about *4,200,000,000,000,000,000* miles.

One solution to the problem of writing these long numerals was the definition of a light year. *A light year is the distance light travels in one year.* It travels about 186,000 miles per second. It is not always convenient, however, to use a light year as a unit of measure.

The atomic scientist must often use very small numbers. One drop of water contains more than 333,000,000 molecules. Each of these molecules contains two hydrogen atoms and one oxygen atom. So an oxygen atom is about $\frac{1}{1,000,000,000}$ of the size of a drop of water. The mass of a hydrogen atom is about $\frac{1}{273,171,800,000,000,000,000,000,000}$ of a pound.

3.14 Let's Explore

1. Consider the number 25,000,000,000,000. Another numeral for this number is ‘$25,000,000,000 \times 10^3$’. Find some other numerals for 25,000,000,000,000 using powers of 10.
2. How many different numerals did you write for exercise one? You should easily write a dozen different numerals for 25,000,000,000,000.
3. Write several numerals for 3,000,000,000,000,000 using exponential notation for a positive power of ten.
4. What is the simplest numeral you can write, using exponential notation for a power of ten, for 10,000,000,000,000,000,000?
5. What is the simplest numeral you can write, using exponential notation for a power of ten, for .000000000000001?
6. What is the simplest numeral you can write, in the same way, for 20,000,000,000,000,000,000? Compare with exercise 4 to get an idea.
7. What is the simplest numeral you can write, in the same way, for .000000000000005? Compare with exercise 5 to get an idea.

*8. Suppose you wished to multiply 4,000,000,000 by 6,000,000,000,000,000.

$$4,000,000,000 = 4 \times 10^?$$

$$6,000,000,000,000,000 = 6 \times 10^?$$

So $4,000,000,000 \times 6,000,000,000,000,000 = (4 \times 10^9) \times (6 \times 10^{15})$.
Using the associative and commutative properties,
$(4 \times 10^9) \times (6 \times 10^{15}) = (4 \times 6) \times (? \times ?) = 24 \times 10^?$

In order to simplify writing numerals for large and for small numbers, a special type of numeral for arithmetic numbers is used. It is called *scientific notation.*

Scientific notation *for a number consists of one of the following:*

A. A decimal numeral for a number between one and ten, a multiplication symbol, and exponential notation for a power of ten.

Example:

$$1.23 \times 10^5$$

or

B. Exponential notation for a power of ten.

Example:

$$10^7$$

or

C. A symbol for a number between 1 and 10.

Example:

$$1.76$$

How do we find scientific notation for a number when we know its decimal numeral?

3.15 Let's Explore

1. Divide 304 by 10^2. Is your result a number between 1 and 10?
2. Divide 4623 by 10^3. Is your result a number between 1 and 10?
3. Is $32653 \times \frac{1}{10^4}$ between 1 and 10?
4. Multiply by a power of ten to get a number between 1 and 10. Copy and complete the table:

462	$462 \times \frac{1}{10^2}$	4.62
3571	$3571 \times \frac{1}{10^3}$	3.571
4763		
19		
347,000		
49,700		
3,470,000		

5. These symbols all represent the multiplicative identity for numbers of arithmetic:

$$\frac{10^2}{10^2}, \qquad \frac{10^5}{10^5}, \qquad \frac{10^3}{10^3}, \quad \text{etc.}$$

Copy and complete this table. (It will help to use exercise 4.)

462	$462 \times \frac{10^2}{10^2}$	$\left(462 \times \frac{1}{10^2}\right) \times 10^2$	4.62×10^2
3571	$3571 \times \frac{10^3}{10^3}$	$\left(3571 \times \frac{1}{10^3}\right) \times 10^3$	3.571×10^3
4763			
19			
347,000			
49,700			
3,470,000			

In exercise 5 above you wrote, in the last column, scientific notation for the number named in the first column. How did you do it? You multiplied by 1. You chose a symbol for 1 like $\frac{\text{‘}10^4\text{’}}{10^4}$. The exponent, in this case 4, tells the number of places to the left to move the decimal point, to get a name for a number between 1 and 10.

EXERCISES

Write scientific notation for these numbers:

1. 472
2. 3256
3. 32,000,000
4. 8,234,000,000,000,000
5. 6,502,000,000
6. 37,400,000,000
7. 895,200,000,000,000
8. 45,600,000,000,000,000,000

We know how to write scientific notation for large numbers (greater than 1). How do we write scientific notation for small numbers (less than 1)?

Archimedes

287-212 B.C.

Archimedes lived in the ancient city of Syracuse in Sicily. There he discovered some of the basic principles of mechanics, among them the law of the lever. He is reputed to have said "Give me a rod long enough and a place to stand, and I will move the earth." His work in geometry was influenced by his knowledge of mechanics, for he thought of balancing triangles, and other figures. With this approach he discovered many important properties.

In an essay he showed how the numeration system of his time could be used to name the number of grains of sand in the universe. Here, probably for the first time, numerals for very large numbers were invented. The largest number used was 10^{63}.

3.16 Let's Explore

1. Multiply .024 by 10^2. Is your result a number between 1 and 10?
2. Multiply .00035 by 10^4. Is your result a number between 1 and 10?
3. Is $.00000813 \times 10^7$ a number between 1 and 10?
4. Multiply by a power of ten to get a number between 1 and 10. Copy and complete the table:

.015	$.015 \times 10^2$	1.5
.00241	$.00241 \times 10^3$	2.41
.0000583		
.000104		
.0000000782		
.000004138		

5. These symbols all represent the multiplicative identity for numbers of arithmetic:

$$\frac{10^3}{10^3}, \qquad 10^3 \times 10^{-3}, \qquad \frac{10^6}{10^6}, \qquad 10^6 \times 10^{-6}$$

Copy and complete this table. (It will help to use exercise 4.)

.015	$.015\times(10^2\times10^{-2})$	$(.015\times10^2)\times10^{-2}$	1.5×10^{-2}
.00241	$.00241\times(10^3\times10^{-3})$	$(.00241\times10^3)\times10^{-3}$	2.41×10^{-3}
.0000583			
.000104			
.0000000782			
.000004138			

In exercise 5 above you wrote scientific notation (last column) for the numbers named in the first column. You did it by multiplying by 1.

EXERCISES

1. Write scientific notation for these numbers:
   - a. 100024
   - b. .0456
   - c. .00000001468
   - d. .34567
   - e. .00000000004579
   - f. .00000000000062

2. Write scientific notation for these numbers:
   - a. 430,000
   - b. .000142
   - c. 38,500,000
   - d. .0000005821
   - e. .000000000000402
   - f. 818,000,000,000,000,000

3. Multiply. Write scientific notation for each answer. (Hint: Write scientific notation for each factor. Then use the commutative and associative properties.)

Example:

	$4{,}000{,}000 \times 52{,}000{,}000{,}000$
(Write scientific notation.)	$(4 \times 10^{6}) \times (5.2 \times 10^{10})$
(Use commutative and associative properties.)	$(4 \times 5.2) \times (10^{6} \times 10^{10})$
	20.8×10^{16}
(Multiply by 1.)	$20.8 \times (10^{-1} \times 10) \times 10^{16}$
(Use associative property.)	$(20.8 \times 10^{-1}) \times (10 \times 10^{16})$
	2.08×10^{17}

- a. 5,000,000,000 × 8,000,000,000,000
- b. 25,000,000,000 × 160,000,000,000,000,000
- c. 125,000,000,000,000 × 320,000,000,000,000,000
- d. 6,400,000,000 × 100,000,000 × 9,000,000,000,000,000
- e. 20,000,000 × 15,000,000,000,000,000 × 10,000,000,000

4. Multiply. Write scientific notation for each answer.

Example:

	$.0024 \times 5{,}000{,}000 \times .000001$
(Write scientific notation.)	$(2.4 \times 10^{-3}) \times (5 \times 10^{6}) \times 10^{-6}$
(Use commutative and associative properties.)	$(2.4 \times 5) \times (10^{-3} \times 10^{6} \times 10^{-6})$
	12×10^{-3}
(Multiply by 1.)	$12 \times (10^{-1} \times 10) \times 10^{-3}$
(Use associative property.)	$(12 \times 10^{-1}) \times (10 \times 10^{-3})$
	1.2×10^{-2}

- a. .005 × .0008
- b. .00016 × 2000
- c. 55,000,000,000 × 10,000,000 × .0000009
- d. 3,718,000,000 × 2,000,000,000 × .0000000005
- e. 83,000 × .000005 × 9,000,000
- f. .00000007 × .0000000000048
- g. .0000062 × .0000000012 × .00000001

h. 4,982,000,000 × 9,360,000,000
i. .00000342 × 513,000,000,000
j. 34,570,000,000,000,000 × 14,200,000,000
k. .00000051 × .00342
l. 381,000,000,000 × .000000000000456
m. .000000021 × 85,000,000,000,000
n. 8,900,000,000 × .00042
o. 45,000,000,000 × 270,000
p. .000000872 × .00000543
q. 4,753,000,000 × 42,000,000,000
r. .0000473 × .000000892
s. 35,000,000,000 × .00000071
t. .000000342 × .0300000000

5. Find the speed of light in feet per second. (Remember, 1 mile = 5280 ft.) Write scientific notation for your answer.

*6. Write scientific notation for the number of pounds in a uranium atom. (An atom of uranium has a mass 238.07 times as great as an atom of hydrogen.)

CHANGE OF METRIC UNITS

Do you remember that the metric system of measures is based on decimal (base ten) numeration? The units of length, for example, are related as follows:

1 kilometer (km) = 10^3 meters (m)
1 hectometer (hm) = 10^2 m
1 dekameter (dkm) = 10 m
1 decimeter (dm) = 10^{-1} m
1 centimeter (cm) = 10^{-2} m
1 millimeter (mm) = 10^{-3} m

From this list it appears as if scientific notation might be especially useful with the metric system. That is true, because to change units, we multiply by powers of ten.

There are two commonly used metric prefixes not shown in the above list. They are *mega*, meaning 10^6, or 1 million, and *micro*, meaning 10^{-6}, or 1 millionth. Thus 1 megameter = 10^6 meter. You might expect we would call 10^{-6} meter a *micrometer*. We do not, because there is a measuring instrument by that name. Instead we call 10^{-6} meter a *micron,* and abbreviate it 'μ'. This symbol is the Greek letter *mu.* A *micro*gram is of course 10^{-6} grams, or 1 millionth of a gram. 'Microgram' is abbreviated 'μg'.

Here are some examples of changing metric units. Notice that scientific notation does help simplify.

Example 1:

Change 3.25 kilometers to meters.

$$3.25 \text{ km} \cdot \frac{10^3 \text{ m}}{1 \text{ km}} = 3.25 \times 10^3 \frac{\text{km}}{\text{km}} \text{ m}$$

$$\text{or } 3.25 \times 10^3 \text{ m}$$

Example 2:

Change 25.7 hectometers to centimeters.

$$25.7 \text{ hm} \cdot \frac{10^2 \text{ m}}{1 \text{ hm}} \cdot \frac{1 \text{ cm}}{10^{-2}\text{m}} =$$

$$25.7 \cdot \frac{10^2}{10^{-2}} \frac{\text{hm}}{\text{hm}} \frac{\text{m}}{\text{m}} \text{ cm} =$$

$$25.7 \times 10^4 \text{ cm} =$$

$$2.57 \times 10^5 \text{ cm, or } 257{,}000 \text{ cm}$$

EXERCISES

1. Make a table for metric mass units. Include megagrams and micrograms.

2. a. In radio and television people talk about kilocycles and megacycles. Guess what these words mean. Then look them up to see if you were right.
   b. What is a microsecond?

3. Make these changes of metric units. Write scientific notation for your answers.

a. 35.72 dkm	change to m
b. 306 mm	change to m
c. 4,020,000μ	change to m
d. 3.21×10^3 km	change to m
e. 8.92 hm	change to mm
f. 75.3 cm	change to km

g. 3.26×10^{-5} m change to μ
h. 43.2 megameters change to km
i. 27.2 kg change to g
j. 3.51 kg change to μg
k. 475 μg change to kg
l. 2.75 mg change to kg

*4. Make these changes of compound metric units. Write scientific notation for your answers.

Example:

$$\text{Change } 35\,\frac{\text{m}}{\text{sec}} \text{ to } \frac{\text{km}}{\text{hr}}$$

$$35\,\frac{\text{m}}{\text{sec}} \cdot \frac{1\ \text{km}}{10^3\ \text{m}} \cdot \frac{60\ \text{sec}}{1\ \text{min}} \cdot \frac{60\ \text{min}}{1\ \text{hr}} =$$

$$35 \times 60 \times 60 \times 10^{-3}\,\frac{\text{m}}{\text{m}}\,\frac{\text{sec}}{\text{sec}}\,\frac{\text{min}}{\text{min}}\,\frac{\text{km}}{\text{hr}} =$$

$$1.26 \times 10^2\,\frac{\text{km}}{\text{hr}}$$

a. $\frac{1}{18}\,\frac{\text{m}}{\text{sec}}$ change to $\frac{\text{km}}{\text{hr}}$
b. $9\,\frac{\text{km}}{\text{hr}}$ change to $\frac{\text{m}}{\text{sec}}$
c. $\frac{1}{6}\,\frac{\text{hm}}{\text{sec}}$ change to $\frac{\text{m}}{\text{min}}$
d. $6.25\,\frac{\text{dkm}}{\text{sec}}$ change to $\frac{\text{dm}}{\text{min}}$
e. $13.7\,\frac{\text{mg}}{\text{cm}}$ change to $\frac{\text{kg}}{\text{hm}}$
f. $4.6 \times 10^{-5}\,\frac{\text{mg}}{\text{mm}}$ change to $\frac{\text{kg}}{\text{m}}$
g. $3.2 \times 10^{4}\,\frac{\text{m}}{\text{sec}}$ change to $\frac{\text{km}}{\text{hr}}$

APPROXIMATING WITH SCIENTIFIC NOTATION

Whenever you estimate you are making an *approximation*. There are other kinds of approximation, too. Whenever we find a number close to another one we are approximating. Sometimes when we approximate we say we are *rounding* a number. You have probably done this before, but let's make sure we understand.

3.17 Let's Explore

Use a number line for these exercises if it will help you.

1. 1.3 comes between what two whole numbers? Is it closer to 1 or to 2?
2. 52 comes between what two multiples of 10? Is it closer to 50 or to 60?
3. 389 comes between what two multiples of 100? Is it closer to 300 or to 400?
4. 389 comes between what two multiples of 10? Is it closer to 380 or 390?
5. 4891.3 comes between what two multiples of 1000? To which is it closer?
6. 4891.3 comes between what two multiples of 100? To which is it closer?
7. 4891.3 comes between what two multiples of 10? To which is it closer?
8. 4891.3 comes between what two integers? To which is it closer?

In exercise 5 above you "rounded" 4891.3 to the nearest thousand, in exercise 6 you "rounded" 4891.3 to the nearest hundred, etc.

If you were to round 365 to the nearest hundred you should write '400'. However, if you were to round 365 to the nearest ten the answer is not so obvious. The number 365 is halfway between 360 and 370. It is not closer to either 360 or 370. A special rule for rounding a number which is halfway between the two numbers is needed. We shall make the following agreement:

If the number is halfway between the two numbers being considered, round so that the last non-zero digit is even.

Examples:

Rounded to hundreds,
4246 would be 4200
5294 would be 5300
14,350 would be 14,400
450 would be 400
2359 would be 2400

Rounded to tens,
45 would be 40
155 would be 160
165 would be 160

EXERCISES

1. Round the following to the nearest 1000:
   - a. 25,340
   - b. 5931
   - c. 1,423,810
   - d. 2500
   - e. 3500
2. Round the following to the nearest 10:
   - a. 2583
   - b. 3645
   - c. 12,855
   - d. 3402
   - e. 24
3. Round the following to the nearest hundredth:
   - a. 346.2091
   - b. 1.347
   - c. .2459
   - d. .015
   - e. 4.045

Scientific notation for 4246 is '4.246×10^3'. When we rounded 4246 to the nearest hundred, we got 4200. Scientific notation for 4200 is '4.2×10^3.'

We can use scientific notation to show how a number is rounded. For example, consider:

36,018.2

If we round to the nearest ten, we get 36,020. Scientific notation '3.602×10^4,' shows that the number was rounded to the nearest ten. The following table shows scientific notation for other ways of rounding 36,018.2.

Rounded to nearest	Decimal numeral	Scientific notation
unit	36,018	3.6018×10^4
ten	36,020	3.602×10^4
hundred	36,000	3.60×10^4
thousand	36,000	3.6×10^4
ten thousand	40,000	4×10^4

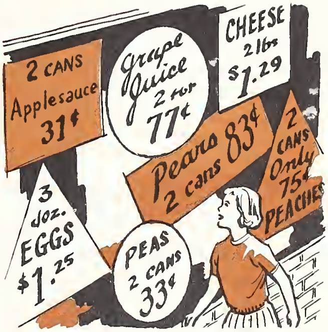
EXERCISES

1. Write scientific notation to show 470,608 rounded to the nearest:
   - a. unit
   - b. ten
   - c. hundred
   - d. thousand
   - e. ten thousand
   - f. hundred thousand

2. Write scientific notation to show 3806.107 rounded to the nearest:
   a. thousandth
   b. hundredth
   c. tenth
   d. unit
   e. ten
   f. hundred
   g. thousand

ROUNDING TO THE NEAREST CENT

We do not always follow the same rules for rounding numbers. In the case of money, some special cases must be considered.



3.18 Let's Explore

1. If you go to the grocery store and buy a can of peaches marked "2 cans for 75¢," how much do you pay? Can you explain why this is necessary?
2. If you buy 5 cans of peas marked "2 cans for 33¢", how much do you pay? As the cashier figures your bill, she might say, "33¢, 33¢, and 17¢." Explain.
3. Suppose you borrow \$133 at $6\frac{1}{2}\%$ interest for 1 year. The interest charged you would be \$133 × $6\frac{1}{2}\%$ × 1. How much interest would you pay? Do you think the bank would drop the part of a cent?
4. Suppose you have \$225 in a savings account. The bank pays interest at the rate of $3\frac{1}{2}\%$ per annum. How much interest would you receive for the year? (Multiply \$225 × $3\frac{1}{2}\%$ × 1.) Does the bank pay you the extra half a cent?

In the case of rounding numbers for money, round off any part of a cent to the next highest cent except on income from investment. In figuring income from investments, drop any part of a cent.

EXERCISES

1. Mary bought the following items at the grocery store:
   3 lbs. of cheese at 2 lbs. for \$1.29
   1 can of pears at 2 cans for \$.83
   1 bottle of grape juice at 2 bottles for \$.77
   3 cans of applesauce at 2 cans for \$.31

   Find the amount of her bill.

2. How much did Sue pay for 2 dozen eggs priced 3 dozen for $1.25?

3. Bill bought four golf balls. The price of the balls was $10 per dozen. How much did Bill pay?

4. How much interest did Mr. Smith pay on a $525 loan if the bank charges $5\frac{1}{2}\%$ interest per annum and he repaid the money at the end of six months?

5. Mrs. Thomas has $962.14 in her savings account. Her bank pays 3% interest per annum. How much interest does she receive for the year?

Let's Estimate

Now that you are familiar with scientific notation you can use it in estimating. You will find it most useful in complicated problems. In many cases you will need to use pencil and paper, even though you are estimating.

Example 1:

$$\begin{aligned} 3{,}012{,}000 \times 4121 &\approx 3 \times 10^6 \times 4 \times 10^3 \\ &\approx 12 \times 10^9 \text{ or, } 1.2 \times 10^{10} \end{aligned}$$

Example 2:

$$\begin{aligned} \frac{5673}{.231} &\approx \frac{5.7 \times 10^3}{2.3 \times 10^{-1}} \approx \frac{5.7}{2.3} \times 10^4 \\ &\approx 2 \times 10^4 \end{aligned}$$

Example 3:

$$\frac{3012 \times .0321 \times 485}{6143 \times 6.11} \approx \frac{3 \times 10^3 \times 3 \times 10^{-2} \times 5 \times 10^2}{6 \times 10^3 \times 6}$$

$$\approx \frac{3 \times 3 \times 5}{6 \times 6} \times \frac{10^3 \times 10^{-2} \times 10^2}{10^3}$$

$$\approx \tfrac{1}{2} \times \tfrac{1}{2} \times 5 \times \frac{10^3}{10^3} \approx \tfrac{5}{4}, \text{ or about } 1.25$$

Try these. Use pencil and paper only if you need to.

1. $4{,}142{,}000 \times 5143$
2. $38{,}514 \times 2412$
3. $9{,}416{,}000 \times 356$
4. $750{,}100 \times 1023$
5. $315 \times .0051$
6. $412 \times .00024$
7. $3{,}014{,}000 \times .0000072$
8. $.0000052 \times 7{,}043{,}000$
9. $.000213 \times .0478$
10. $.000028 \times .502$
11. $\dfrac{3816}{7.46}$
12. $\dfrac{8492}{41.7}$
13. $\dfrac{47{,}812}{6.45}$
14. $\dfrac{97{,}444}{12.8}$
15. $\dfrac{5917}{.076}$
16. $\dfrac{3106}{.0059}$
17. $\dfrac{.064}{3.40}$
18. $\dfrac{.0039}{212}$
19. $\dfrac{617 \times .014}{3.18}$
20. $\dfrac{413 \times .0051}{65.9}$
21. $\dfrac{9562 \times .46 \times 216}{3111 \times 5.12}$
22. $\dfrac{4814 \times .0031 \times 7.2}{861 \times 17.6}$
23. $\dfrac{.0125 \times 39{,}412}{375 \times 4112 \times .142}$
24. $\dfrac{.0034 \times 673}{1240 \times 317 \times 14.7}$

DO YOU KNOW THESE TERMS?

To show that you know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

power (113)
base (113)
exponent (113)
exponential notation (113)
light year (143)
scientific notation (144)
megameter (150)
micro (150)
micron (151)
approximation (152)
rounding a number (152)

CHAPTER SUMMARY

Powers of numbers (113)
Multiplying powers (114)
Dividing using exponential notation (118)
Negative integers as exponents (120)
Expanded numerals (130)
A power of a power (131)
Multiplication of integers (137)
Multiplying and dividing by powers of 10 (140)
Scientific notation (143)
Change of metric units (150)
Approximating with scientific notation (152)
Rounding to the nearest cent (155)

REVIEW EXERCISES

1. Write exponential notation for each of the following:
   - a. $4 \cdot 4 \cdot 4$
   - b. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$
   - c. $3 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 3$
   - d. $\dfrac{5 \cdot 5 \cdot 5 \cdot 2 \cdot 3 \cdot 3}{5 \cdot 2 \cdot 2}$
   - e. $\dfrac{a \cdot a \cdot b \cdot b \cdot a \cdot b \cdot a \cdot a}{a \cdot b \cdot b \cdot b \cdot b \cdot b \cdot a}$
   - f. $5^2 \cdot 5^4$
   - g. $\dfrac{7^3}{7}$
   - h. $\dfrac{10^5 \cdot 10^7}{10^6}$
   - i. $5^2 \cdot 5^{^-3} \cdot 5^{^-2} \cdot 5^3 \cdot 5^7$
   - j. $9^7 \div 9^{^-3}$
   - k. $10^5 \div 10^2$
   - l. $\dfrac{x^3 \cdot x^7 \cdot x^{^-8}}{x^{^-3} \cdot x^5}$

2. Simplify the following:
   - a. ${}^-5 + {}^-8 \neq 16$
   - b. ${}^-5 \cdot {}^-8 \cdot 16$
   - c. $({}^-5 \cdot {}^-8) \div 16$
   - d. ${}^-a + {}^-a + 5a$
   - e. ${}^-b + a + b + {}^-a$

3. Write an expanded numeral for each of the following (use negative exponents where appropriate):
   a. 3421_5
   b. 256.34_{10}
   c. 25.793_{10}
   d. 1110.1111_2
   e. 3678.928763_{10}

4. Write prime factorizations for each of the following:
   a. 1250
   b. 324
   c. 1089
   d. 369
   e. 96

5. Write exponential notation for each of the numbers in exercise 4.

6. Write exponential notation for each of the following:
   a. $(5^2)^3$
   b. $(10^{-3})^4$
   c. $(a^{-2})^{-7}$
   d. $(ab^2)^5$
   e. $(a^{-2}bc^5)^{-4}$

7. Find the following products:
   a. $.52 \times 1000$
   b. 5.2×1000
   c. $.0052 \times 10$
   d. $.052 \times 100$
   e. 520×1000

8. Find the following quotients:
   a. $.52 \div 1000$
   b. $5.2 \div 1000$
   c. $.0052 \div 10$
   d. $.052 \div 100$
   e. $520 \div 1000$

9. Write scientific notation for each of the following:
   a. 52,340,000,000
   b. .0000000032
   c. 4,700,000,000,000,000
   d. 32.03
   e. .000000000000000000549

10. Round each of the following numbers as indicated.
    a. 4567 to nearest thousand
    b. 52,346 to nearest ten
    c. 32.489 to nearest tenth
    d. .00034 to nearest ten-thousandth
    e. 3,478,936,542 to nearest billion

11. How much interest is due on a \$1000 loan for 6 months at $5\frac{3}{4}\%$?

12. John has \$250 in his savings account. The bank pays $3\frac{1}{2}\%$ interest, compounded semiannually. If John does not deposit or withdraw any money for two years, how much money will be in his account?

13. The home arts classes are making tablecloths. Find the cost of the material for one tablecloth, 6′ long if the material sells for 79¢ a yard and is the exact width of the tablecloth. There is a 3% sales tax.

14. "Mary is 110 years old and in grade 21. Her brother is 121 years old and in grade 102. He will graduate next year."

    The paragraph above is sensible when you discover what base numerals are used. How would it read using base 10 numerals?

15. Tell what property of arithmetic numbers is illustrated by each of the following:
    - a. $\frac{3}{4} + \frac{2}{3} = \frac{2}{3} + \frac{3}{4}$
    - b. $\frac{7}{8} \cdot \frac{8}{7} = 1$
    - c. $(\frac{2}{3} \cdot \frac{3}{4}) \cdot \frac{5}{8} = \frac{2}{3} \cdot (\frac{3}{4} \cdot \frac{5}{8})$
    - d. $\frac{1}{2}(\frac{3}{4} + \frac{2}{5}) = (\frac{1}{2} \cdot \frac{3}{4}) + (\frac{1}{2} \cdot \frac{2}{5})$
    - e. $\frac{15}{9} + 0 = \frac{15}{9}$

16. Tell what property of the system of integers is illustrated by each of the following:
    - a. ${}^-3 + 3 = 0$
    - b. ${}^-2 + ({}^-4 + {}^+7) = ({}^-2 + {}^-4) + {}^+7$
    - c. ${}^-7 + 0 = {}^-7$
    - d. ${}^-3 + 9 = 9 + {}^-3$
    - e. $a + b = c$

17. Use a ruler and compass to construct a rectangle having length and width as follows:

    length

    width

18. If flowers are to be planted six inches apart, how many plants are needed to go around three sides of a square-shaped flower garden, eight feet on a side?

19. Write the simplest fractional numeral for each of the following:
    - a. $\frac{9}{6}$
    - b. $\frac{45}{600}$
    - c. $\frac{15 \times 16}{25}$
    - d. $\frac{\frac{3}{4} \times \frac{2}{9}}{\frac{5}{8} \times \frac{2}{3}}$
    - e. $\frac{\frac{1}{2} + \frac{3}{4}}{\frac{7}{8}}$
    - f. 1.8333 · · ·
    - g. .342342
    - h. .242424 · · ·
    - i. 5.151555 · · ·
    - j. 3.4783478

20. Sixty-six inches is how many meters?

READING LIST

Bakst, A. *Mathematics, Its Magic and Mastery.* New York: D. Van Nostrand Co., Inc., 1952. pp. 231-244.

Meyer, J. S. *Fun With Mathematics.* New York: World Book Company, 1952. pp. 71-76.

Ruchlis, H. and Engelhardt, J. *The Story of Mathematics.* Irvington-on-Hudson, N. Y.: Harvey House, Publishers, 1958. pp. 114-116.

University of Maryland Mathematics Project. *Mathematics for the Junior High School, First Book.* College Park: Univ. of Md., 1959. pp. 245-256.

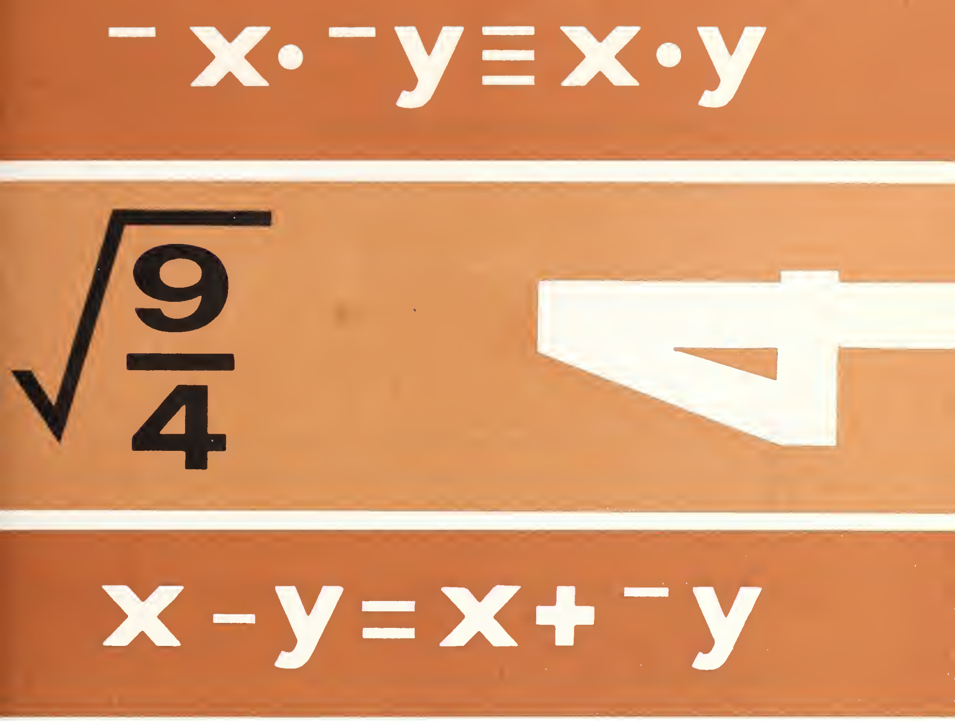
$^{-}1 \cdot n = {}^{-}n$

$x \cdot y = {}^{-}x \cdot {}^{-}y$

4

CHAPTER FOUR

The System of Rational Numbers



INTRODUCTION

We have studied several number systems—the system of whole numbers, the number system of arithmetic, and the system of integers. In the number system of arithmetic every number except zero has a multiplicative inverse (reciprocal). The only number having an additive inverse is zero. In the system of integers all numbers have additive inverses, but only the numbers one and negative one have a multiplicative inverse.

Let's try to make up a better and more complete number system, having both additive inverses and multiplicative inverses. How can we do this? There are several different ways we might try. Let's try the same approach we used for integers. There we started with the whole numbers. For every non-zero whole number we made up an additive inverse.

THE SET OF RATIONAL NUMBERS

We start with the numbers of arithmetic. Remember, an arithmetic number is any number which can be named by a fractional numeral with a whole number for numerator and a natural number for denominator. The only arithmetic number with an additive inverse is 0. For each non-zero number, let's make up an additive inverse. Then our set will contain:

1. *An additive identity—zero.*
2. *All other numbers of arithmetic—such as* $\frac{1}{2}$, *3,* $\frac{17}{11}$, *3.71, etc.*
3. *Additive inverses for all numbers of arithmetic (including zero)—such as* $^{-}\frac{1}{2}$, *⁻3,* $^{-}\frac{3}{4}$, *⁻3.71, etc.*

We shall call all numbers in our new system "rational" numbers. The word *rational* comes from the word *ratio*.

EXERCISES

1. An arithmetic number is a number which can be named by a fractional numeral, like '$\frac{3}{5}$', with a whole number for numerator and a natural number for denominator. To show that a number is a number of arithmetic, we find such a name.

   Show that these are arithmetic numbers:

   a. 1.5
   b. $3 + \frac{1}{2}$
   c. 75%
   d. .04
   e. $5 - 5$
   f. $\frac{4.2}{2.1}$

2. In the set of rational numbers, what is the additive inverse of:

   a. 3.2
   b. 0
   c. $^{-}\frac{5}{4}$
   d. ⁻6.13
   e. ⁻37%
   f. 412%

ORDER AND THE NUMBER LINE

We know how to make a number ray for the numbers of arithmetic. We also know how to make a number line for the integers. Our new system includes the numbers of arithmetic, and the integers too. To make a number line for rational numbers, we can combine the ideas of these other systems.

4.1 Let's Explore

Draw a number line for integers. Use a unit $\frac{1}{2}$ inch long. Label the points.

1. Mark and label points for these numbers on your number line.
   a. $\frac{1}{2}$
   b. $\frac{3}{4}$
   c. $\frac{3}{2}$
   d. 3.3
   e. 4.1
   f. $2 + \frac{2}{3}$

2. Let's agree that the point for $^-\frac{1}{2}$ will be halfway between the points for 0 and $^-1$.
   Mark and label points for $^-\frac{1}{2}$ and $^-\frac{3}{4}$.

3. Mark and label points for these rational numbers.
   a. $^-\frac{1}{4}$
   b. $^-\frac{3}{2}$
   c. $^-250\%$
   d. $^-3.7$
   e. $^-(2 + \frac{3}{4})$
   f. $^-1.1$

We have agreed on one way to make a rational number line. Now we know how to place this set of numbers in order, and can show it on a horizontal number line. We say that one rational number is greater than another if its point on the line is farther to the right. Notice that the zero point separates the line. There is a half-line to the right of it and a half-line to the left of it.

Numbers whose points are on the right half-line are called *positive.* Those whose points are on the left half-line are called *negative.*

We can define *absolute value* the way we did for integers. That is, $|x| = x$ when $x \geq 0$, and $|x| = {}^-x$ when $x < 0$.

Do you remember that the absolute value of an integer tells how far its graph is from the origin? The same is true for the absolute value of a rational number. The point for $^-\frac{3}{2}$, for example, is $|^-\frac{3}{2}|$, or $\frac{3}{2}$, units from the origin.

EXERCISES

1. Simplify.
   a. $|3|$
   b. $|^-3|$
   c. $^-|1.2|$
   d. $^-|^-\frac{2}{3}|$
   e. $|^-0|$
   f. $^-|35\%|$

2. Which of these sentences are true?
   a. $^-2 < 3.4$
   b. $^-5 > {}^-2$

c. $\frac{3}{2} > \frac{1}{4}$
d. ${}^-\frac{3}{2} > {}^-\frac{1}{4}$
e. ${}^-3.5 \leq {}^-1.2$
f. $1.2 \leq 3.5$
g. ${}^-\frac{3}{5} < {}^-\frac{2}{5}$
h. ${}^-\frac{3}{5} < \frac{2}{5}$
i. $|{}^-6.51| \geq 3.2$
j. ${}^-|{}^-12\%| = {}^-12\%$

3. Find decimal numerals for these rational numbers.
a. ${}^-\frac{3}{4}$
b. $\frac{5}{8}$
c. ${}^-\frac{4}{5}$
d. ${}^-\frac{2}{3}$
e. $\frac{9}{7}$
f. ${}^-\frac{25}{4}$
g. $\frac{17}{3}$
h. ${}^-\frac{24}{11}$

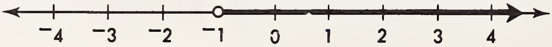
4. Find fractional numerals for these rational numbers.
a. .5
b. ${}^-1.21$
c. 17%
d. ${}^-3.5\%$
e. ${}^-.3333 \cdots$
f. $.9999 \cdots$

5. Find base five numerals for these rational numbers.
a. 16
b. ${}^-12$
c. ${}^-76$
d. ${}^-32$
e. 153
f. ${}^-158$

6. Use a rational number line. Plot graphs of these sentences and describe them geometrically. (Hint: Some of these will consist of all the points on a ray, half-line, or segment. Remember that we show a ray like this: ●————→ A half line like this: ○————→ Mark your graphs as shown in the following examples.)

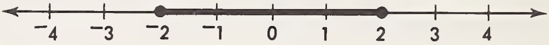
Example 1:

The graph of $x > {}^-1$ does not include the point for ${}^-1$, but includes all points for larger numbers (all points to the right of ${}^-1$). The graph is therefore this half-line:



Example 2:

The graph of $|x| \leq 2$ contains the points for 2 and ${}^-2$. It does not contain points for 3 or ${}^-3$ because they make the sentence false. It does contain points for 1 and ${}^-1$. It contains the origin. In fact it contains all points between 2 and ${}^-2$. The graph is therefore this segment:



a. $y = {}^{-}\frac{1}{2}$

b. $x < 2.1$

c. $y = {}^{-}4.3$

d. $t \geq {}^{-}1.3$

e. $y \leq {}^{-}2.2$

f. $|x| = \frac{4}{3}$

g. $y > {}^{-}\frac{3}{2}$

h. $|y| \leq 1.5$

i. $|x| = 0.3$

j. $|t| < \frac{8}{3}$

k. $|y| \leq \frac{2}{3}$

l. $|x| > \frac{5}{3}$

m. $|y| < \frac{9}{5}$

n. $|y| \geq \frac{5}{3}$

ADDITION AND SUBTRACTION

How shall we define addition for rational numbers? We already know how to add positive rational numbers because they are also numbers of arithmetic. Zero is the additive identity for numbers of arithmetic, so we know how to add zero and any non-negative rational number. A rational number plus its additive inverse is zero, by definition of additive inverses. For example,

$$\frac{2}{3} + {}^{-}\frac{2}{3} = 0, \qquad {}^{-}\frac{12}{5} + \frac{12}{5} = 0$$

A definition of addition is needed for the remaining cases:

1. A negative number and zero.
2. Two negative numbers.
3. A positive number and a negative number with different absolute values. (If they have the same absolute value they are additive inverses of each other.)

Do you remember the definition of addition for integers? There were these very same cases. We wanted the commutative and associative properties to hold, and made definitions accordingly. We proved:

1. *If* the commutative and associative properties are to hold, *then* we had to make definitions as we did.
2. *If* we make definitions as we did, *then* the commutative and associative properties hold.

Since the rational numbers are so similar, let's try a definition like the one for integers.

1. *For any number of arithmetic* $\boldsymbol{n}$*,*

$$0 + {}^{-}n = {}^{-}n \quad \text{and} \quad {}^{-}n + 0 = {}^{-}n.$$

2. *For any numbers of arithmetic* $\boldsymbol{m}$ *and* $\boldsymbol{n}$*,*

$${}^{-}m + {}^{-}n = {}^{-}(m + n).$$

3. *If* $\boldsymbol{m}$ *and* $\boldsymbol{n}$ *are two numbers of arithmetic,*

   a. ${}^{-}m + n = {}^{-}(m - n)$ *and*
   $n + {}^{-}m = {}^{-}(m - n)$*, when* $m > n$.

   b. ${}^{-}m + n = n - m$ *and*
   $n + {}^{-}m = n - m$*, when* $m < n$.

Does this definition give us the properties we want? Is the set of rational numbers closed under addition? Is addition an operation? Is it commutative? Is it associative? Is there an identity? Does every number have an additive inverse? The answer is "yes" to all of these questions, but we shall not take time to prove them now. You may wish to do these proofs for yourself.

EXERCISES

1. Do these additions of rational numbers. Write the simplest fractional numeral for your result.

a. $\frac{2}{3} + \frac{2}{3}$
b. $\frac{3}{7} + \frac{2}{7}$
c. $^{-}\frac{3}{4} + {}^{-}\frac{2}{4}$
d. $^{-}\frac{3}{5} + {}^{-}\frac{1}{5}$
e. $\frac{3}{7} + {}^{-}\frac{1}{7}$
f. $^{+}\frac{9}{5} + {}^{-}\frac{2}{5}$
g. $^{-}\frac{2}{9} + \frac{13}{9}$
h. $^{-}\frac{7}{12} + \frac{11}{12}$
i. $\frac{2}{13} + {}^{-}\frac{5}{13}$
j. $^{+}\frac{4}{9} + {}^{-}\frac{7}{9}$
k. $^{-}\frac{5}{8} + \frac{2}{8}$
l. $^{-}\frac{8}{5} + \frac{2}{5}$

2. Do these additions of rational numbers. Write the simplest decimal numeral for your result.

a. $3.5 + 1.2$
b. $^{+}4.6 + {}^{-}3.1$
c. $17.45 + {}^{-}15.34$
d. $.012 + {}^{-}.011$
e. $^{-}271 + 275$
f. $^{-}3712 + {}^{+}4713$
g. $31.6 + {}^{-}32.7$
h. $1.032 + {}^{-}2.144$
i. $^{-}12.03 + 11.01$
j. $^{-}141.32 + 121.21$
k. $^{-}4.32 + {}^{-}2.14$
l. $^{-}.014 + {}^{-}.998$
m. $13.5\% + .112$
n. $^{-}18.2\% + {}^{-}.035$
o. $^{-}1.2 \times 10^{3} + 1.4 \times 10^{3}$
p. $^{-}7.8 \times 10^{-2} + {}^{-}1.1 \times 10^{-2}$

3. What is the additive inverse of:

a. $^{-}\frac{2}{3}$
b. 0
c. 1.34
d. $^{-}.0031$
e. $|{}^{-}3|$
f. $^{-}|0|$
g. $|{}^{-}4 + 2|$
h. $|3.2 + {}^{-}1.2|$

4. Which of these sentences are true?

a. $\frac{5}{9} + \frac{2}{9} = \frac{7}{9}$
b. $\frac{3}{5} + \frac{1}{5} = \frac{3}{5}$
c. $\frac{2}{3} + \frac{4}{3} < \frac{1}{2} + \frac{3}{2}$
d. $^{-}\frac{9}{5} + \frac{4}{5} \geq \frac{6}{3} - 1$
e. $^{-}1.23 + 4.56 = {}^{-}1.11 + {}^{-}2.22$
f. $|{}^{-}1.5 + 3.5| < |8.61 + {}^{-}7.01|$
g. $^{-}4.1 + 2.6 \neq 2.6 + {}^{-}4.1$
h. $^{-}19.1 + {}^{-}173.6 = {}^{-}173.6 + {}^{-}19.1$
i. $(1.2 + {}^{-}3.1) + {}^{-}65.7 < 1.2 + ({}^{-}3.1 + {}^{-}65.7)$
j. $({}^{-}17.6 + {}^{-}12.1) + {}^{-}31.82 = {}^{-}17.6 + ({}^{-}12.1 + {}^{-}31.82)$

SUBTRACTION

Subtraction is the opposite of addition. For integers, we defined subtraction as adding an additive inverse. For rational numbers we can do the same.

Example:

To subtract $\frac{2}{3} - \frac{3}{4}$, we can add $\frac{2}{3} + {}^{-}\frac{3}{4}$.
To subtract $\frac{9}{7} - {}^{-}\frac{13}{12}$, we can add $\frac{9}{7} + \frac{13}{12}$.

Number phrases can be used to tell how to subtract. We use variables such as 'x' and 'y' which can be replaced by symbols for *any* rational numbers. Then the subtraction is shown by the phrase '$x - y$'. The additive inverse of y is represented by the phrase '${}^{-}y$'. To show the addition we use the phrase '$x + {}^{-}y$'. The following sentence then tells how we subtract. It is the definition of subtraction.

For any rational numbers $\boldsymbol{x}$, $\boldsymbol{y}$,

$$x - y = x + {}^{-}y.$$

EXERCISES

1. Do these subtractions. Write the simplest fractional numeral for your result.

a. $\frac{5}{4} - \frac{3}{4}$
b. ${}^{+}\frac{2}{3} - {}^{-}\frac{4}{3}$
c. ${}^{-}\frac{3}{8} - \frac{7}{8}$
d. ${}^{-}\frac{8}{9} - {}^{-}\frac{5}{9}$
e. ${}^{-}\frac{4}{5} - \frac{3}{5}$
f. $\frac{5}{12} - \frac{6}{12}$
g. $\frac{4}{15} - {}^{-}\frac{7}{15}$
h. ${}^{-}\frac{19}{7} - {}^{-}\frac{6}{7}$
i. ${}^{+}\frac{18}{11} - {}^{-}\frac{5}{11}$
j. ${}^{-}\frac{13}{23} - {}^{-}\frac{6}{23}$
k. ${}^{-}\frac{7}{5} - {}^{-}\frac{7}{5}$
l. ${}^{-}\frac{7}{5} - \frac{7}{5}$

2. Do these subtractions. Write the simplest decimal numeral for your result.

a. $324 - 123$
b. $671 - {}^{-}123$
c. $13.6 - 12.4$
d. $1.83 - {}^{-}2.12$
e. $141.3 - 145.5$
f. ${}^{-}.012 - .403$
g. ${}^{-}1.36 - {}^{-}14.01$
h. ${}^{+}17.853 - {}^{-}3.023$
i. ${}^{-}34.1 - 12.3$
j. $.165 - 14.1\%$
k. ${}^{-}1.23\% - .105$
l. $35.1\% - {}^{-}12.2\%$
m. ${}^{-}250\% - {}^{-}37.5\%$
n. ${}^{+}127\% - {}^{+}17\%$

3. Solve these equations. They refer to the system of rational numbers.
   a. $\frac{3}{4} + \square = \frac{1}{2}$
   b. ${}^{-}8.2 - p = 9.3$
   c. ${}^{-}1.3 = x - 1.2$
   d. ${}^{-}\frac{5}{9} + \frac{2}{9} = {}^{-}t$
   e. $3.1 = 1.9 - {}^{-}y$
   f. ${}^{-}z + \frac{5}{7} = \frac{9}{7}$
   g. ${}^{-}2.3 - y = {}^{-}4.2$
   h. $q - 37.2\% = 1.11$
   i. ${}^{-}16.8\% - {}^{-}b = {}^{-}5.1\%$

4. We have defined subtraction of rational numbers. We add the additive inverse of the subtrahend.
   When we subtracted in the number system of arithmetic, we did not do this. Do we get the same results?
   Do these subtractions two ways and compare. Do them first the way you did in the system of arithmetic. Then do them by adding an inverse.
   a. $\frac{5}{3} - \frac{2}{3}$
   b. $18.6 - 7.3$
   c. $\frac{5}{9} - \frac{2}{9}$
   d. $37.82 - 15.69$
   e. $\frac{17}{24} - \frac{5}{12}$
   f. $.0382 - .0046$
   g. $\frac{19}{36} - \frac{2}{9}$

MULTIPLICATION

We know how to multiply in the number system of arithmetic, so we know how to multiply rational numbers in some cases. For the other cases a definition of multiplication is needed. The other cases are:

1. The product of two negative numbers.
2. The product of a positive number and a negative number.
3. The product of zero and a negative number.

If possible we shall keep the familiar properties. How shall we make the definition?

Remember your study of exponents. You found that $(m^2)^3 = m^6$. You discovered that you could multiply exponents in cases like this, where the exponents are whole numbers. You also found that $(m^{{}^{-}2})^3 = m^{{}^{-}6}$. Multiplication for integers had not yet been defined, and this made us realize a useful way to define it. We said that ${}^{-}2 \cdot 3 = {}^{-}6$. Multiplication of integers was defined this way, so that it would be useful with exponents.

We also noticed that $(m^{{}^{-}3})^{{}^{-}4} = m^{12}$, and this led us to define multiplication so that ${}^{-}3 \cdot {}^{-}4 = 12$.

Perhaps we can define multiplication of rational numbers in the same way. Remember we would like our new system to have the commutative, associative, and distributive properties, if possible. Before we make the definition, let's think about these properties.

*4.2 Let's Explore

Let's *imagine* that the commutative, associative, and distributive properties hold for rational numbers. Use these properties on

$$3 \cdot 2 + {}^{-}3 \cdot 2 + {}^{-}3 \cdot {}^{-}2.$$

1. Use the distributive property to factor, like this:

$$(\qquad) \cdot 2 + {}^{-}3 \cdot {}^{-}2$$

2. What number did you name in the parentheses?
3. What is the product of that number and 2?
4. When this number is added to ${}^{-}3 \cdot {}^{-}2$, what do you get? Why?
5. Think again about:

$$3 \cdot 2 + {}^{-}3 \cdot 2 + {}^{-}3 \cdot {}^{-}2$$

Again, *imagine* the familiar properties hold. Now use the distributive property to factor like this:

$$3 \cdot 2 + {}^{-}3 \cdot (\qquad)$$

6. What number did you name in the parentheses?
7. What is the product of that number and ${}^{-}3$?
8. When this number is added to $3 \cdot 2$, what do you get? Why?
9. By *assuming* (imagining) that the distributive property holds, you showed two things about

'$3 \cdot 2 + {}^{-}3 \cdot 2 + {}^{-}3 \cdot {}^{-}2$'.

i. '$3 \cdot 2$' must name the same number.
ii. ______________ must also name the same number.

10. If we want the distributive property in our new system, how must we define '${}^{-}3 \cdot {}^{-}2$'?
11. Think about '$\frac{3}{4} \cdot \frac{3}{5} + {}^{-}\frac{3}{4} \cdot \frac{3}{5} + {}^{-}\frac{3}{4} \cdot {}^{-}\frac{3}{5}$'. Assume the distributive property as before. Show how '${}^{-}\frac{3}{4} \cdot {}^{-}\frac{3}{5}$' must be defined.
12. Think about '$a \cdot b + {}^{-}a \cdot b + {}^{-}a \cdot {}^{-}b$', where '$a$' and '$b$' represent any non-zero numbers of arithmetic. Assume the distributive property, as before. Show how '${}^{-}a \cdot {}^{-}b$' must then be defined.

You have discovered how multiplication of two negative rational numbers must be defined if we want the distributive property for the system. Does this fit what we would like for exponents? Let's make the definition accordingly.

For any numbers of arithmetic $\boldsymbol{m}$ *and* $\boldsymbol{n}$, $\boldsymbol{{}^{-}m \cdot {}^{-}n = m \cdot n}$.

We could also have said to multiply absolute values, or ${}^{-}m \cdot {}^{-}n = |{}^{-}m| \cdot |{}^{-}n|$. The important thing to remember is that the product of two negative numbers is positive.

Now how about the product of a positive and a negative number?

*4.3 Let's Explore

Let's think again about

'$3 \cdot 2 + {}^{-}3 \cdot 2 + {}^{-}3 \cdot {}^{-}2$'.

Assuming the familiar properties, you have already shown that the symbol names the number $3 \cdot 2$.

1. If $3 \cdot 2 + \square = 3 \cdot 2$ is to be true, what number must be named in the box?
2. If $3 \cdot 2 + ({}^{-}3 \cdot 2 + {}^{-}3 \cdot {}^{-}2) = 3 \cdot 2$ is true, what number is named in the parentheses?
3. Is '${}^{-}3 \cdot {}^{-}2$' a name for $3 \cdot 2$? Why?
4. If so, replace '${}^{-}3 \cdot {}^{-}2$' in exercise 2 by '$3 \cdot 2$'.
5. What number is now named in the parentheses?
6. Then is it true that ${}^{-}3 \cdot 2 + 3 \cdot 2 = 0$?
7. Then ${}^{-}3 \cdot 2$ must be the additive __________ of $3 \cdot 2$.
8. Is '${}^{-}(3 \cdot 2)$' a symbol for the additive inverse of $3 \cdot 2$?
9. Is the following sentence true? Why?

   $${}^{-}3 \cdot 2 = {}^{-}(3 \cdot 2)$$
10. If we want the distributive property in our new system, how must be define '${}^{-}3 \cdot 2$'?
11. Think about '$\frac{3}{4} \cdot \frac{3}{5} + {}^{-}\frac{3}{4} \cdot \frac{3}{5} + {}^{-}\frac{3}{4} \cdot {}^{-}\frac{3}{5}$'. Assume the distributive property as before. Show how '${}^{-}\frac{3}{4} \cdot \frac{3}{5}$' must then be defined.
12. Think about '$a \cdot b + {}^{-}a \cdot b + {}^{-}a \cdot {}^{-}b$', where '$a$' and '$b$' represent any non-zero numbers of arithmetic. Assume the distributive property as before. Show how '${}^{-}a \cdot b$' must then be defined.

Hieronimo Cardano

1501-1576

Cardano lived in 16th Century Italy. They were days of intrigue, plots and jealous accusations. He was typical of his day. The first signs of the theory of probability are found in his handbook for gamblers.

Mathematicians of his day were greatly interested in solving equations, and in proving they knew more than their colleagues by presenting them with problems to solve that used discoveries they themselves had made. Cardano invented some ingenious problems for his rivals, involving magic squares, circumnavigation of the globe, and problems of pursuit.

In his work, *Ars Magna*, he organized, and added to, the algebra known at that time. His rivals accused him of including their work in his book, as well as his own. In this work Cardano recognizes that the product of two negative numbers must be positive.

You have discovered how multiplication of a negative and a positive rational number must be defined if our new system is to have the distributive property. Let's make the definition accordingly.

For any numbers of arithmetic, $\boldsymbol{m}$ *and* $\boldsymbol{n}$*,*
$^{-}\boldsymbol{m} \cdot \boldsymbol{n} = {}^{-}(\boldsymbol{m} \cdot \boldsymbol{n})$. *Also,*
$\boldsymbol{n} \cdot {}^{-}\boldsymbol{m} = {}^{-}(\boldsymbol{m} \cdot \boldsymbol{n})$.

We could also have said to multiply absolute values and then take the additive inverse of the result. That is, $^{-}m \cdot n = {}^{-}(|{}^{-}m| \cdot |n|)$. An important fact to remember is that the product of a positive and a negative rational number is negative.

To make multiplication by zero the same as in earlier systems, we should agree that the product of 0 and any rational number is zero.

For any rational number $\boldsymbol{x}$*,*
$\boldsymbol{x} \cdot 0 = 0$ *and* $0 \cdot \boldsymbol{x} = 0$.

EXERCISES

1. Multiply.
   - a. $3 \cdot {}^{-}2$
   - b. $^{-}\frac{2}{3} \cdot \frac{5}{3}$
   - c. $^{-}\frac{1}{2} \cdot {}^{-}\frac{3}{4}$
   - d. $^{-}\frac{5}{9} \cdot {}^{-}\frac{2}{7}$
   - e. $({}^{-}4 \cdot \frac{1}{2}) \cdot {}^{-}\frac{2}{3}$
   - f. $^{-}\frac{3}{2} \cdot ({}^{-}\frac{1}{5} \cdot {}^{-}\frac{3}{4})$
   - g. $({}^{-}3 \cdot {}^{-}\frac{2}{5}) \cdot \frac{1}{5}$
   - h. $({}^{-}3 \cdot {}^{-}2) \cdot {}^{-}\frac{1}{2}$
   - i. $(\frac{1}{2} \cdot {}^{-}\frac{3}{4}) \cdot \frac{2}{3}$
   - j. $\frac{1}{2} \cdot ({}^{-}\frac{3}{4} \cdot \frac{2}{3})$
   - k. $({}^{-}\frac{7}{3} \cdot {}^{-}\frac{1}{2}) \cdot {}^{-}\frac{5}{2}$
   - l. $^{-}\frac{7}{3} \cdot ({}^{-}\frac{1}{2} \cdot {}^{-}\frac{5}{2})$
   - m. $(\frac{2}{5} \cdot {}^{-}\frac{3}{7}) \cdot ({}^{-}\frac{1}{4} \cdot {}^{-}\frac{3}{2})$
   - n. $({}^{-}2 \cdot {}^{-}\frac{2}{3}) \cdot ({}^{-}4 \cdot {}^{-}\frac{1}{5})$
   - o. $0 \cdot ({}^{-}\frac{3}{2} \cdot \frac{4}{5})$
   - p. $^{-}\frac{4}{3} \cdot (\frac{0}{3} \cdot \frac{5}{4})$
2. Multiply. Write the simplest decimal numeral for your result.
   - a. $^{-}12 \times 3$
   - b. $6 \times {}^{-}7$
   - c. $1.3 \times {}^{-}2.1$
   - d. $^{-}7.1 \times .3$
   - e. $^{-}16.1 \times {}^{-}1.2$
   - f. $^{-}.031 \times {}^{-}1.2$

g. $13\% \times {}^{-}2$
h. ${}^{-}31.2\% \times 3$
i. ${}^{-}1.2\% \times {}^{-}20\%$
j. ${}^{-}125\% \times {}^{-}10\%$
k. ${}^{-}13_5 \times 21_5$
l. ${}^{-}23 \times {}^{-}11$
m. $(1.2 \times {}^{-}30) \times {}^{-}3$
n. ${}^{-}2.2 \times ({}^{-}3.1 \times {}^{-}10)$

3. For numbers of arithmetic, scientific notation usually consists of a decimal numeral for a number between 1 and 10, a multiplication sign and a symbol for a power of ten. For rational numbers it is the same except that for negative numbers we must write a minus sign.
Multiply and write scientific notation for your result.
a. $({}^{-}1.3 \times 10^3) \times (2.1 \times 10^5)$
b. $({}^{-}8.2 \times 10^2) \times ({}^{-}4.1 \times 10^7)$
c. $(4.5 \times 10^{{}^{-}3}) \times (1.1 \times 10^6)$
d. $(6.2 \times 10^{{}^{-}5}) \times ({}^{-}2.1 \times 10^{{}^{-}7})$
e. $({}^{-}6.1 \times 10^{{}^{-}7}) \times ({}^{-}3.1 \times 10^3)$
f. $(4.4 \times 10^6) \times ({}^{-}1.1 \times 10^{{}^{-}10})$

We have defined multiplication. We defined it as we did because we wanted the familiar properties to hold. This does not mean that they necessarily do. What properties does the new system have?

*4.4 Let's Explore

1. Check to see whether multiplication is an operation. Check these cases. The results must be unique.
   a. Multiplying two positive rational numbers.
   b. Multiplying any number by zero.
   c. Multiplying a negative and a positive number.
   d. Multiplying two negative numbers.

2. Check to see whether the set of rational numbers is closed under multiplication. Check the same cases as in exercise 1.

3. Check to see whether multiplication is commutative. Check the same cases as in exercises 1 and 2.

4. Now let's check the associative property. There are several cases. In each case, we see whether:

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

Case 1. All three numbers are non-negative.
Case 2. At least one of the numbers is zero, the others are positive, negative, or zero.
Case 3. Some or all of the numbers are negative, the rest are positive.
Before checking Case 3, try these examples to get an idea:

a. ${}^{-}2 \cdot (3 \cdot 5)$ a′ $({}^{-}2 \cdot 3) \cdot 5$
b. $4 \cdot ({}^{-}\frac{1}{2} \cdot 3)$ b′ $(4 \cdot {}^{-}\frac{1}{2}) \cdot 3$

c. $\frac{1}{3} \cdot (2 \cdot {}^-6)$	c′ $(\frac{1}{3} \cdot 2) \cdot {}^-6$
d. ${}^-2 \cdot ({}^-3 \cdot 5)$	d′ $({}^-2 \cdot {}^-3) \cdot 5$
e. ${}^-4 \cdot (\frac{1}{2} \cdot {}^-3)$	e′ $({}^-4 \cdot \frac{1}{2}) \cdot {}^-3$
f. $4 \cdot ({}^-\frac{1}{2} \cdot {}^-3)$	f′ $(4 \cdot {}^-\frac{1}{2}) \cdot {}^-3$
g. ${}^-4 \cdot ({}^-\frac{1}{2} \cdot 3)$	g′ $({}^-4 \cdot {}^-\frac{1}{2}) \cdot 3$
h. ${}^-2 \cdot ({}^-3 \cdot {}^-5)$	h′ $({}^-2 \cdot {}^-3) \cdot {}^-5$
i. ${}^-4 \cdot ({}^-\frac{1}{2} \cdot {}^-3)$	i′ $({}^-4 \cdot {}^-\frac{1}{2}) \cdot {}^-3$

You have probably concluded that multiplication of rational numbers is an operation; also that there is closure, and that the commutative and associative properties hold. If you did you are correct.

In the system of rational numbers:
Multiplication is an operation (results are unique).
The set of rational numbers is closed under multiplication.
Multiplication is commutative.
Multiplication is associative.

A MULTIPLICATIVE IDENTITY

In the number system of arithmetic, the number 1 is the multiplicative identity. Does the system of rational numbers have a multiplicative identity?

Suppose m is a positive rational number. This means it is also a number of arithmetic. Then

$$1 \cdot m = m,$$

because 1 is the multiplicative identity in the number system of arithmetic.

We also know that:

$$1 \cdot 0 = 0$$

Why?

Suppose a negative rational number, ${}^-m$, is multiplied by 1. By the definition of multiplication:

$$1 \cdot {}^-m = {}^-(1 \cdot m) = {}^-m$$

There are no other possibilities. A rational number is either positive, zero or negative. Therefore we have just proved that the number 1 is the multiplicative identity.

The number 1 is the multiplicative identity in the system of rational numbers.
For any rational number x, $1 \cdot x = x$.

THE NUMBER $^{-}1$

Suppose a rational number is multiplied by $^{-}1$. What is the result?

4.5 Let's Explore

1. Multiply, using the definition of multiplication for a positive and a negative number.
    - a. $^{-}1 \cdot 3$
    - b. $^{-}1 \cdot \frac{4}{3}$
    - c. $^{-}1 \cdot \frac{5}{9}$
    - d. $^{-}1 \cdot 7$
    - e. $^{-}1 \cdot 4.6$
    - f. $^{-}1 \cdot 31.2$
2. Multiply, using the definition of multiplication for two negative numbers.
    - a. $^{-}1 \cdot {}^{-}5$
    - b. $^{-}1 \cdot {}^{-}\frac{3}{3}$
    - c. $^{-}1 \cdot {}^{-}\frac{5}{9}$
    - d. $^{-}1 \cdot {}^{-}8$
    - e. $^{-}1 \cdot {}^{-}3.1$
    - f. $^{-}1 \cdot {}^{-}7.12$
3. a. What is the product of $^{-}1 \cdot 0$?
   b. What is $^{-}0$?
4. When a rational number (positive, negative or zero) is multiplied by $^{-}1$, the result is the ____________ ____________ of that number.

You have now seen that multiplying a rational number by $^{-}1$ gives its additive inverse. You will find this fact important, and should remember it.

For any rational number x,
$$^{-}1 \cdot x = {}^{-}x.$$

MULTIPLICATIVE INVERSES AND DIVISION

There is a multiplicative identity in the system of rational numbers, so it is sensible to ask which numbers have multiplicative inverses.

All positive rational numbers have multiplicative inverses, or reciprocals, because they are numbers of arithmetic. Zero has no reciprocal because $0 \cdot x$ can never be 1 for any x. The product of 0 and any rational number is zero. Do negative rational numbers have reciprocals?

4.6 Let's Explore

1. Multiply:
   a. $\frac{3}{2} \times \frac{2}{3}$
   b. $^-\frac{3}{4} \cdot \frac{4}{3}$
   c. $^-\frac{7}{5} \times {}^-\frac{5}{7}$
   d. $^-\frac{5}{1} \cdot {}^-\frac{1}{5}$
   e. $^-\frac{13}{12} \cdot \frac{12}{13}$
   f. $^-\frac{13}{12} \cdot {}^-\frac{12}{13}$

2. If ‘$\frac{m}{n}$’ represents any positive rational number, what does ‘$\frac{n}{m}$’ represent?

3. If ‘$\frac{^-m}{n}$’ represents any negative rational number, what does ‘$\frac{^-n}{m}$’ represent?

4. Multiply:
   a. $2 \times .5$
   b. $4 \times .25$
   c. $^-.2 \times {}^-5$
   d. $^-10 \times {}^-.1$
   e. $10^2 \times 10^{^-2}$
   f. $^-10^5 \times {}^-10^{^-5}$

We now know that every non-zero rational number (negative or positive) has a multiplicative inverse.

If ‘$\frac{m}{n}$’ *represents a positive rational number, then* ‘$\frac{n}{m}$’ *represents its reciprocal.*

If ‘$\frac{^-m}{n}$’ *represents a negative rational number, then* ‘$\frac{^-n}{m}$’ *represents its reciprocal.*

You have already seen that the reciprocal of a rational number can be named in more than one way. Later we shall find still other ways of naming reciprocals.

Since all non-zero rational numbers have multiplicative inverses, we can always divide, except by zero. To divide a rational number by another, we multiply by the reciprocal of the second. For example:

$$\frac{3}{4} \div {}^-\frac{2}{3} = \frac{3}{4} \cdot {}^-\frac{3}{2}$$

or

$$\frac{\frac{3}{4}}{{}^-\frac{2}{3}} = \frac{3}{4} \cdot {}^-\frac{3}{2}$$

EXERCISES

1. Name the multiplicative inverse of each of these numbers.

a. $\frac{1}{2}$
b. 2
c. $\frac{3}{4}$
d. ${}^-\frac{3}{4}$
e. ${}^-\frac{1}{3}$
f. ${}^-3$
g. $\frac{3}{7}$
h. ${}^-\frac{5}{9}$
i. $\frac{8}{5}$
j. ${}^-\frac{9}{4}$
k. 0
l. ${}^-5$

2. Divide:

a. $\dfrac{\frac{4}{5}}{\frac{3}{7}}$
b. $\dfrac{{}^-\frac{3}{4}}{{}^-\frac{5}{7}}$
c. $\dfrac{\frac{4}{5}}{{}^-\frac{7}{12}}$
d. $\dfrac{{}^-\frac{12}{10}}{{}^-\frac{13}{7}}$
e. $\dfrac{{}^-\frac{16}{9}}{\frac{5}{4}}$
f. $\dfrac{\frac{20}{5}}{{}^-\frac{17}{8}}$
g. $\dfrac{{}^-\frac{42}{3}}{{}^-\frac{5}{7}}$
h. $\dfrac{{}^-12}{\frac{5}{8}}$
i. $\dfrac{{}^-5}{{}^-7}$
j. $\dfrac{{}^-3}{4}$

Reciprocals of rational numbers can be named in several ways, as we have already seen. There is another way of naming them that is sometimes important.

4.7 Let's Explore

1. Divide:

a. $\dfrac{1}{\frac{2}{3}}$
b. $\dfrac{1}{{}^-\frac{3}{2}}$
c. $\dfrac{1}{{}^-\frac{5}{8}}$
d. $\dfrac{1}{\frac{8}{5}}$
e. $\dfrac{1}{{}^-2}$
f. $\dfrac{1}{\frac{8}{7}}$
g. $\dfrac{1}{{}^-\frac{4}{5}}$
h. $\dfrac{1}{{}^-\frac{3}{7}}$

2. If 'x' represents any non-zero rational number, then $\frac{\text{'1'}}{x}$ represents its ______________________________.

You should have discovered that when 1 is divided by a non-zero rational number, the result is the reciprocal of that number. Therefore $\frac{\text{'1'}}{x}$ represents the reciprocal of x.

If 'x' represents any non-zero rational number, then $\frac{\text{'1'}}{x}$ represents its reciprocal, or multiplicative inverse.

EXERCISES

1. Name the reciprocal of each of these numbers.

a. $\frac{1}{\frac{2}{3}}$

b. $\frac{1}{^{-}\frac{4}{5}}$

c. $\frac{1}{\frac{7}{8}}$

d. $\frac{1}{^{-}\frac{4}{7}}$

e. 3

f. $^{-}5$

g. $^{-}7$

h. 1.374

i. $^{-}4.621$

j. $^{-}3\frac{1}{2}$

2. Which of these sentences are true?

a. $\frac{1}{\frac{2}{3}} \cdot \frac{2}{3} = 1$

b. $^{-}\frac{3}{3} \cdot \frac{1}{\frac{3}{4}} = 1$

c. $4.3 \cdot \frac{1}{4.3} = 1$

d. $^{-}\frac{6}{5} \cdot \frac{1}{\frac{6}{5}} = {}^{-}1$

e. $\frac{1}{^{-}5\%} \cdot 5\% = 1$

f. $\frac{7}{8} \cdot \frac{1}{\frac{8}{7}} = 1$

USING THE NUMBER 1

Remember how the number 1 was used in arithmetic. It can be used the same way with the system of rational numbers. The number 1 can be named by symbols like the following:

$$\frac{\frac{2}{3}}{\frac{2}{3}}, \qquad \frac{{}^{-}\frac{3}{5}}{{}^{-}\frac{3}{5}}, \qquad \frac{{}^{-}1}{{}^{-}1}.$$

The number 1 can be named by any fractional numeral whose numerator and denominator are the same (not zero). Why?

In the system of arithmetic the number 1 was used to find different fractional names of numbers, as in these examples:

$$\frac{2}{3} = \frac{2}{3} \times \frac{5}{5} = \frac{10}{15}$$

$$\frac{16}{12} = \frac{2 \cdot 2 \cdot 2 \cdot 2}{2 \cdot 2 \cdot 3} = \frac{2 \cdot 2}{2 \cdot 2} \times \frac{2 \cdot 2}{3} = \frac{4}{3}$$

In multiplication the number 1 was used to help in simplifying, like this:

$$\frac{2}{3} \times \frac{7}{5} \times \frac{9}{4} = \frac{2 \cdot 7 \cdot 9}{3 \cdot 5 \cdot 4} = \frac{2 \cdot 3 \cdot 3 \cdot 7}{2 \cdot 2 \cdot 3 \cdot 5} = \frac{2 \cdot 3}{2 \cdot 3} \times \frac{3 \cdot 7}{2 \cdot 5} = \frac{21}{10}$$

In dividing the number 1 was used, as in the following examples:

$$\frac{\frac{2}{3}}{\frac{4}{5}} = \frac{\frac{2}{3}}{\frac{4}{5}} \times \frac{15}{15} = \frac{10}{12}$$

$$\frac{\frac{5}{7}}{\frac{3}{4}} = \frac{\frac{5}{7}}{\frac{3}{4}} \times \frac{\frac{4}{3}}{\frac{4}{3}} = \frac{\frac{5}{7} \times \frac{4}{3}}{1} = \frac{20}{21}$$

In the first example above we chose ‘$\frac{15}{15}$’ because 15 is the L.C.M. of the denominators.

In the second example above we chose ‘$\frac{\frac{4}{3}}{\frac{4}{3}}$’ so that we would get a denominator of 1 when we multiply. This works because $\frac{4}{3}$ and $\frac{3}{4}$ are multiplicative inverses of each other.

In the system of rational numbers, the number 1 can be used in these same ways.

EXERCISES

1. For each of these numbers, find a fractional numeral with the denominator indicated.

Example:

${}^{-}\frac{3}{5}$, 15.

$${}^{-}\frac{3}{5} = {}^{-}\left(\frac{3}{5} \cdot \frac{3}{3}\right) = {}^{-}\frac{9}{15}$$

a. $\frac{5}{2}$, 16 b. ${}^{-}\frac{10}{3}$, 30 c. ${}^{-}\frac{19}{12}$, 48 d. ${}^{-}\frac{3}{10}$, 100

2. Simplify:

Example:

$$\frac{{}^-4}{6}$$

$$\frac{{}^-4}{6} = {}^-\left(\frac{2 \cdot 2}{3 \cdot 2}\right) = {}^-\left(\frac{2}{3} \cdot \frac{2}{2}\right) = \frac{{}^-2}{3}$$

a. $\frac{5}{15}$ c. $\frac{{}^-14}{30}$ e. $\frac{36}{54}$

b. $\frac{{}^-12}{8}$ d. $\frac{18}{24}$ f. $\frac{{}^-34}{51}$

3. Divide by using the multiplicative identity:

Example:

$$\frac{\frac{5}{9}}{\frac{{}^-4}{7}}$$

$$\frac{\frac{5}{9}}{\frac{{}^-4}{7}} = \frac{\frac{5}{9}}{\frac{{}^-4}{7}} \cdot \frac{\frac{{}^-7}{4}}{\frac{{}^-7}{4}} = \frac{\frac{5}{9} \cdot \frac{{}^-7}{4}}{1} = \frac{{}^-35}{36}$$

a. $\dfrac{\frac{4}{3}}{\frac{{}^-5}{7}}$ d. $\dfrac{\frac{5}{7}}{\frac{4}{3}}$ g. $\dfrac{\frac{{}^-14}{3}}{\frac{{}^-13}{11}}$

b. $\dfrac{\frac{{}^-3}{8}}{\frac{7}{11}}$ e. $\dfrac{\frac{{}^-2}{3}}{\frac{5}{11}}$ h. $\dfrac{\frac{19}{5}}{\frac{27}{7}}$

c. $\dfrac{\frac{{}^-8}{7}}{\frac{{}^-5}{9}}$ f. $\dfrac{\frac{12}{7}}{\frac{{}^-13}{5}}$

4. Divide and simplify, using any method:

a. $\dfrac{\frac{4}{5}}{\frac{{}^-3}{7}}$ e. $\dfrac{\frac{{}^-16}{9}}{\frac{4}{3}}$ i. $\dfrac{{}^-5}{{}^-10}$

b. $\dfrac{\frac{{}^-3}{4}}{\frac{{}^-5}{7}}$ f. $\dfrac{\frac{20}{5}}{\frac{{}^-16}{8}}$ j. $\dfrac{{}^-3}{6}$

c. $\dfrac{\frac{4}{5}}{{}^-\frac{6}{10}}$ g. $\dfrac{\frac{{}^-42}{3}}{\frac{{}^-7}{6}}$

d. $\dfrac{\frac{{}^-12}{10}}{\frac{{}^-24}{5}}$ h. $\dfrac{{}^-12}{\frac{2}{3}}$

MORE ABOUT SYMBOLS AND MULTIPLICATION

We know that ${}^-2 \cdot \frac{5}{3} = \frac{{}^-10}{3}$, by the definition of multiplication. To multiply a positive and a negative number, we multiply their absolute values and then take the additive inverse. What about $2 \cdot \frac{{}^-5}{3}$? The product is also $\frac{{}^-10}{3}$. Then it is true that ${}^-2 \cdot \frac{5}{3} = 2 \cdot \frac{{}^-5}{3}$. Will this work for any such pair of numbers?

4.8 Let's Explore

Multiply and simplify:

1. ${}^{-}\frac{4}{3} \cdot \frac{2}{5}$
2. $\frac{4}{3} \cdot {}^{-}\frac{2}{5}$
3. ${}^{-}\frac{3}{7} \cdot \frac{3}{2}$
4. $\frac{3}{7} \cdot {}^{-}\frac{3}{2}$
5. ${}^{-}\frac{12}{9} \cdot \frac{3}{4}$
6. $\frac{12}{9} \cdot {}^{-}\frac{3}{4}$
7. ${}^{-}7 \cdot 4$
8. $7 \cdot {}^{-}4$

You have seen that the product of a positive and a negative rational number is negative, and that it does not matter which factor is negative. In other words, if 'a' and 'b' represent positive rational numbers, then '${}^{-}a \cdot b$', '$a \cdot {}^{-}b$' and '${}^{-}(a \cdot b)$' all represent the same number.

Suppose the variables could represent *any* rational numbers; positive, negative or zero. Would the same statement hold?

4.9 Let's Explore

1. Let's compare ${}^{-}x \cdot y$, $x \cdot {}^{-}y$ and ${}^{-}(x \cdot y)$.
   a. Suppose $x = {}^{-}2$ and $y = 3$. Then what is ${}^{-}x$, the additive inverse of x?
   b. Multiply 2 by 3 to find ${}^{-}x \cdot y$.
   c. Since $y = 3$, what is its additive inverse, ${}^{-}y$?
   d. Multiply to find $x \cdot {}^{-}y$.
   e. Now find the product $x \cdot y$.
   f. What is ${}^{-}(x \cdot y)$, the additive inverse of $x \cdot y$?
2. Copy and complete this table:

		Find inverse of **x**	Multiply ${}^{-}x$ by y		Find inverse of **y**	Multiply x by ${}^{-}y$		Multiply x by y		Find inverse of $x \cdot y$
x	y	${}^{-}x$	${}^{-}x \cdot y$		${}^{-}y$	$x \cdot {}^{-}y$		$x \cdot y$		${}^{-}(x \cdot y)$
3	${}^{-}2$	${}^{-}3$	${}^{-}3 \cdot {}^{-}2$	6	2	$3 \cdot 2$	6	$3 \cdot {}^{-}2$	${}^{-}6$	6
${}^{-}4$	${}^{-}3$	4	$4 \cdot {}^{-}3$	${}^{-}12$	3	${}^{-}4 \cdot 3$	${}^{-}12$	${}^{-}4 \cdot {}^{-}3$	12	${}^{-}12$
${}^{-}2$	5									
0	${}^{-}2$									
7	0									
0	0									
${}^{-}4$	${}^{-}5$									
6	${}^{-}3$									

You have probably begun to suspect that '$^-x \cdot y$' and '$x \cdot {}^-y$' and '$^-(x \cdot y)$' will all name the same number for *all* rational number replacements. Of course you did not prove this conjecture. How could it be proved?

4.10 Let's Explore

You know that for any rational number x, $^-1 \cdot x = {}^-x$. This is true whether x is positive, negative or zero.

1. Then for any rational numbers x and y,
   $^-x \cdot y = ({}^-1 \cdot \underline{\qquad}) \cdot y$ (fill in the blank).
2. Using the equation of exercise 1 and the associative property of multiplication, we get
   $^-x \cdot y = {}^-1 \cdot (\underline{\quad} \cdot \underline{\quad})$ (fill in the blanks).
3. Now, instead of '$^-1 \cdot$' we may simply write '$^-$', because $^-1 \cdot x = {}^-x$ for all replacements. Rewrite the sentence without the '1' in it.
   You have now proved that $^-x \cdot y = {}^-(x \cdot y)$ for all rational numbers x and y.
4. Now let's look at '$x \cdot {}^-y$'. For any rational numbers x and y, we know that
   $x \cdot {}^-y = x \cdot (\underline{\quad} \cdot y)$ (fill in the blank and give a reason to prove you are right).
5. If the equation of exercise 4 is true for all replacements, then so is the following equation. Why?
   $x \cdot {}^-y = {}^-(x \cdot y)$.

You have now *proved* your conjecture. It is an important fact for later work, so let's remember it.

> *If '**x**' and '**y**' represent **any rational** numbers, then '$^-\mathbf{x} \cdot \mathbf{y}$', '$\mathbf{x} \cdot {}^-\mathbf{y}$', and '$^-(\mathbf{x} \cdot \mathbf{y})$' all represent the same number.*

EXERCISES

1. Simplify:
   a. $^-\frac{3}{5} \cdot \frac{2}{7}$
   b. $\frac{3}{5} \cdot {}^-\frac{2}{7}$

c. ${}^{-}\frac{4}{3} \cdot \frac{2}{5}$
d. ${}^{-}\frac{2}{5} \cdot \frac{4}{3}$
e. ${}^{-}(\frac{3}{5} \cdot \frac{2}{7})$
f. ${}^{-}(\frac{4}{3} \cdot \frac{2}{5})$

2. Simplify:
   a. ${}^{-}\frac{2}{7} \cdot \frac{3}{5}$
   b. ${}^{-}\frac{3}{7} \cdot \frac{2}{5}$
   c. $\frac{2}{7} \cdot {}^{-}\frac{3}{5}$
   d. ${}^{-}\frac{2}{5} \cdot \frac{3}{7}$
   e. ${}^{-}7 \cdot \frac{2}{9}$
   f. ${}^{-}2 \cdot \frac{7}{9}$
   g. $2 \cdot {}^{-}\frac{7}{9}$
   h. $\frac{7}{9} \cdot {}^{-}2$
   i. ${}^{-}\frac{1}{9} \cdot 7 \cdot 2$
   j. ${}^{-}\frac{1}{9} \cdot {}^{-}7 \cdot {}^{-}2$

3. Which of these sentences are true?
   a. $\frac{7}{23} \cdot {}^{-}\frac{35}{12} = {}^{-}\frac{7}{23} \cdot \frac{35}{12}$
   b. $\frac{314}{17} \cdot {}^{-}\frac{9}{23} = {}^{-}\frac{314}{17} \cdot \frac{9}{23}$
   c. $\frac{65}{21} \cdot {}^{-}\frac{13}{4} = {}^{-}(\frac{65}{21} \cdot \frac{13}{4})$
   d. ${}^{-}\frac{416}{29} \cdot \frac{35}{12} = {}^{-}(\frac{416}{29} \cdot \frac{35}{12})$
   e. $2 \cdot {}^{-}\frac{12}{21} = 12 \cdot {}^{-}\frac{2}{21}$
   f. ${}^{-}\frac{4}{3} \cdot 127 = \frac{127}{3} \cdot {}^{-}4$
   g. $\frac{4}{3} \cdot {}^{-}127 = {}^{-}4 \cdot \frac{127}{3}$

4. Which of these sentences are true?
   a. $3 + {}^{-}5 \cdot 2 = 3 + {}^{-}(5 \cdot 2)$
   b. $\frac{7}{2} - 4 \cdot {}^{-}\frac{2}{3} = \frac{7}{2} - {}^{-}(4 \cdot \frac{2}{3})$
   c. ${}^{-}\frac{3}{5} + {}^{-}(6 \cdot 5) = {}^{-}\frac{3}{5} + {}^{-}6 \cdot 5$
   d. $6 - {}^{-}(3 \cdot 4) = 6 + {}^{-}3 \cdot 4$
   e. $x + {}^{-}3 \cdot y = x + {}^{-}(3 \cdot y)$ for all rational numbers x and y.
   f. $2 \cdot x + {}^{-}4 \cdot y = 2 \cdot x - 4 \cdot y$ for all rational numbers x and y.
   g. $x + {}^{-}1 \cdot y = x + {}^{-}y$ for all rational numbers x and y.
   h. $x - (3 \cdot y) = x + {}^{-}3 \cdot y$ for all rational numbers x and y.
   i. $3 \cdot x + ({}^{-}2 \cdot y) = x - 2 \cdot y$ for all rational numbers x and y.

EXPONENTS AND SQUARE ROOTS

In the system of whole numbers and the number system of arithmetic, multiplication was defined. Therefore it was sensible to use exponent symbols and square root symbols. In the system of integers we did not define multiplication (at least at first), so it was not sensible to use exponent and square root symbols. Now let's study the use of these symbols with rational numbers.

4.11 Let's Explore

1. Write exponential notation for $({}^{-}2) \cdot ({}^{-}2) \cdot ({}^{-}2) \cdot ({}^{-}2)$.
2. Write the simplest decimal numeral for the number named in exercise 1.

3. Write exponential notation for $2 \cdot 2 \cdot 2 \cdot 2$.

4. Write a symbol for the additive inverse of the number named in exercise 3. (Continue to use exponential notation.)

5. Write the simplest decimal numeral for the number named in exercise 4.

6. Compare the symbols which you wrote in exercises 1 and 4.

You should have discovered that we must make some agreement about parentheses and minus signs with exponential notation. Is ‘$^{-}2^4$’ to mean the additive inverse of 2^4? Is it to mean the fourth power of $^{-}2$?

Let’s make the same agreement which mathematicians have made: ‘$^{-}2^4$’ will mean the additive inverse of 2^4. The fourth power of $^{-}2$ will be named ‘$(^{-}2)^4$’.

*If ‘**a**’ is any symbol for a rational number, and ‘**b**’ is any exponent symbol, ‘$^{-}a^b$’ means **the additive inverse of** a^b. ‘$(^{-}a)^b$’ means **the bth power of** ^{-}a.*

Now let’s see what we can learn about square roots in the system of rational numbers.

4.12 Let’s Explore

1. Multiply:
   a. $3 \cdot 3$
   b. $(^{-}3) \cdot (^{-}3)$
   c. $\frac{2}{3} \cdot \frac{2}{3}$
   d. $(^{-}\frac{2}{3}) \cdot (^{-}\frac{2}{3})$
   e. $\frac{5}{2} \cdot \frac{5}{2}$
   f. $(^{-}\frac{5}{2}) \cdot (^{-}\frac{5}{2})$
   g. $\frac{3}{7} \cdot \frac{3}{7}$
   h. $(^{-}\frac{3}{7}) \cdot (^{-}\frac{3}{7})$

2. What is a square root of:
   a. 9
   b. $\frac{4}{9}$
   c. $\frac{25}{4}$
   d. $\frac{9}{49}$
   e. 0
   f. $^{-}4$

3. What have you discovered about square roots of negative rational numbers?

4. What have you discovered about square roots of positive rational numbers?

5. What have you discovered about the square root of 0?

You should have discovered that negative rational numbers do not have square roots. Zero has only one square root. Positive rational numbers may have two square roots, one positive and one negative.

Let's consider the radical symbol '$\sqrt{\ }$'. We would prefer that expressions using this symbol name one number, at most. What should '$\sqrt{9}$' mean? Should it mean 3 or ${}^{-}3$? We must decide. Mathematicians have agreed that it should mean the non-negative square root, 3. Let's make the same agreement.

Then how could we name the non-positive square root? We can do it like this: '${}^{-}\sqrt{9}$'. This symbol means *the additive inverse of* $\sqrt{9}$.

If 'R' represents a non-negative rational number, then '$\sqrt{R}$' names the non-negative square root of R, if there is one. The symbol '${}^{-}\sqrt{R}$' names the non-positive square root of R if there is one.

If 'R' represents any negative rational number, then '$\sqrt{R}$' and '${}^{-}\sqrt{R}$' do not name any rational number.

EXERCISES

1. Write exponential notation for these numbers:
   - a. $({}^{-}3) \cdot ({}^{-}3) \cdot ({}^{-}3) \cdot ({}^{-}3)$
   - b. $({}^{-}4) \cdot ({}^{-}4)$
   - c. $({}^{-}7) \cdot ({}^{-}7) \cdot ({}^{-}7)$
   - d. $({}^{-}5) \cdot ({}^{-}5) \cdot ({}^{-}5) \cdot ({}^{-}5) \cdot ({}^{-}5)$

2. Write decimal numerals for these numbers:
   - a. $({}^{-}3)^2$
   - b. ${}^{-}3^2$
   - c. $({}^{-}3)^4$
   - d. ${}^{-}3^4$
   - e. $({}^{-}5)^3$
   - f. ${}^{-}5^3$
   - g. $({}^{-}2)^4$
   - h. ${}^{-}2^4$
   - i. $({}^{-}4)^3$
   - j. ${}^{-}4^3$
   - k. $({}^{-}2)^5$
   - l. ${}^{-}2^5$
   - m. $({}^{-}5)^4$
   - n. ${}^{-}5^4$

3. Tell which of these symbols do not name rational numbers. Simplify each of the others:

a. $\sqrt{16}$
b. ${}^{-}\sqrt{16}$
c. $\sqrt{{}^{-}5}$
d. ${}^{-}\sqrt{{}^{-}5}$
e. $\sqrt{2 - \frac{6}{3}}$
f. ${}^{-}\sqrt{2 - \frac{6}{3}}$
g. $\sqrt{\frac{9}{25}}$
h. ${}^{-}\sqrt{\frac{9}{25}}$
i. $\sqrt{{}^{-}\frac{36}{25}}$
j. ${}^{-}\sqrt{{}^{-}\frac{36}{25}}$
k. $\sqrt{\frac{0}{36}}$
l. $\sqrt{{}^{-}\frac{0}{36}}$
m. ${}^{-}\sqrt{\frac{0}{36}}$
n. ${}^{-}\sqrt{{}^{-}\frac{0}{36}}$
o. $\sqrt{\frac{100}{144}}$
p. ${}^{-}\sqrt{\frac{100}{144}}$

4. Simplify:

a. $\sqrt{21 + 4}$
b. $\sqrt{31 + 5}$
c. ${}^{-}\sqrt{47 + 2}$
d. ${}^{-}\sqrt{17 - 1}$
e. $\sqrt{25 + 11}$
f. $\sqrt{16 + 9}$
g. $\sqrt{16} + \sqrt{9}$
h. $\sqrt{64 + 36}$
i. $\sqrt{64} + \sqrt{36}$
j. $\sqrt{36 - 11}$
k. $\sqrt{100 - 64}$
l. $\sqrt{100} - \sqrt{64}$

5. Write scientific notation for each of these numbers:

Examples:

$${}^{-}3710 = {}^{-}(3.71 \times 10^{3}), \text{ or } {}^{-}3.71 \times 10^{3}$$
$${}^{-}.0025 = {}^{-}(2.5 \times 10^{{}^{-}3}), \text{ or } {}^{-}2.5 \times 10^{{}^{-}3}$$

a. $^{-}47{,}520$
b. $^{-}382{,}000$
c. $^{-}.000031$
d. $^{-}.000000827$

THE DISTRIBUTIVE PROPERTY

We know the properties of addition and multiplication in the system of rational numbers. Are the two operations related by the distributive property? We know that for the non-negative rational numbers the distributive property holds. Why? Does it hold when some of the numbers are negative?

*4.13 Let's Explore

1. Simplify:

   a. ${}^{-}3 \cdot (2 + \frac{1}{3})$
   b. $\frac{1}{2} \cdot ({}^{-}4 + 6)$
   c. $({}^{-}6 + {}^{-}9) \cdot \frac{1}{3}$
   d. ${}^{-}\frac{2}{5} \cdot ({}^{-}10 + 5)$
   e. $({}^{-}12 + {}^{-}\frac{3}{2}) \cdot {}^{-}\frac{2}{3}$

   a′. ${}^{-}3 \cdot 2 + {}^{-}3 \cdot \frac{1}{3}$
   b′. $\frac{1}{2} \cdot {}^{-}4 + \frac{1}{2} \cdot 6$
   c′. ${}^{-}6 \cdot \frac{1}{3} + {}^{-}9 \cdot \frac{1}{3}$
   d′. ${}^{-}\frac{2}{5} \cdot {}^{-}10 + {}^{-}\frac{2}{5} \cdot 5$
   e′. ${}^{-}12 \cdot {}^{-}\frac{2}{3} + {}^{-}\frac{3}{2} \cdot {}^{-}\frac{2}{3}$

   Does it *look* as though the rational numbers have the distributive property?

2. Let's think about this product ${}^{-}3 \cdot (2 + \frac{1}{3})$. It is the product of a negative and a positive number. How did we define multiplication in this case?

3. If we multiply $|{}^{-}3|$ by $|2 + \frac{1}{3}|$, we are multiplying positive numbers—numbers of arithmetic. Do we have the distributive property in this case?

4. Can you now prove that $x \cdot (y + z) = x \cdot y + x \cdot z$, whenever x is negative and y, z are positive?

5. Suppose x is positive and y, z are negative. Can you prove the distributive property holds?

6. How many other cases are there? List them.

7. See if you can prove that the distributive property holds in each case.

You have probably proved that the distributive property holds for all rational numbers. This is an important fact.

The distributive property of multiplication over addition holds in the system of rational numbers.
For any rational numbers $\mathbf{x}$*,* $\mathbf{y}$ *and* $\mathbf{z}$*,* $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$*.*

EXERCISES

1. Simplify:

   a. ${}^{-}3 \cdot \frac{5}{9} \cdot \frac{6}{10}$
   b. ${}^{-}\frac{3}{5} \cdot {}^{-}\frac{10}{7} \cdot \frac{14}{3}$
   c. ${}^{-}\frac{6}{11} \cdot {}^{-}\frac{10}{7} \cdot \frac{22}{3}$

   d. ${}^{-}\frac{5}{3} \cdot {}^{-}\frac{4}{7} \cdot {}^{-}\frac{3}{10} \cdot {}^{-}\frac{14}{9}$
   e. ${}^{-}\frac{17}{3} \cdot \frac{4}{5} \cdot {}^{-}\frac{6}{17}$
   f. $\frac{5}{9} \cdot {}^{-}\frac{7}{3} \cdot {}^{-}\frac{9}{5} \cdot \frac{3}{7}$

2. Use the distributive property to write other names for these rational numbers.

Example:

$$^{-}\tfrac{3}{2} \cdot (\tfrac{5}{7} + {}^{-}\tfrac{1}{7}) = {}^{-}\tfrac{3}{2} \cdot \tfrac{5}{7} + {}^{-}\tfrac{3}{2} \cdot {}^{-}\tfrac{1}{7}$$

a. $\frac{2}{3} \cdot ({}^{-}\frac{1}{3} + \frac{5}{7})$

b. ${}^{-}\frac{9}{5} \cdot (\frac{2}{7} + {}^{-}\frac{1}{4})$

c. $1.3 \cdot ({}^{-}4.1 + {}^{-}2.5)$

d. ${}^{-}\frac{4}{3} \cdot (12 + {}^{-}4.6)$

3. Factor:

Example:

$$\tfrac{1}{2} \cdot {}^{-}\tfrac{1}{4} + \tfrac{1}{2} \cdot \tfrac{3}{8} = \tfrac{1}{2} \cdot ({}^{-}\tfrac{1}{4} + \tfrac{3}{8})$$

a. ${}^{-}\frac{3}{4} \cdot \frac{1}{5} + {}^{-}\frac{3}{4} \cdot \frac{5}{2}$

b. ${}^{-}\frac{11}{2} \cdot \frac{4}{5} + \frac{13}{12} \cdot \frac{4}{5}$

c. $1.2 \times \frac{2}{3} + {}^{-}4.8 \times \frac{2}{3}$

d. $8.6 \times {}^{-}4.3 + {}^{-}4.3 \times 2.1$

e. $\frac{18}{5} \cdot \frac{2}{7} + \frac{18}{5} \cdot {}^{-}\frac{3}{10} + \frac{18}{5} \cdot \frac{1}{7}$

f. ${}^{-}4.6 \times 1.5 + 3.2 \times {}^{-}4.6 + {}^{-}4.6 \times .01$

g. $\frac{3}{2} \cdot {}^{-}\frac{1}{5} + {}^{-}\frac{3}{8} \cdot \frac{1}{5}$

h. $\frac{3}{8} \times \frac{9}{5} + {}^{-}\frac{3}{8} \times \frac{11}{12}$

i. $14.6 \times \frac{3}{11} + {}^{-}14.6 \times \frac{4}{13}$

j. $3.61 \times 4.8 + {}^{-}3.61 \times {}^{-}1.4$

Now that we know rational numbers have the distributive property of multiplication over addition we may use it to help us divide in certain cases. These are cases in which the dividend is thought of as being a sum of several numbers.

EXERCISES

1. Divide:

Example:

$$\begin{aligned}(4 + \tfrac{1}{3}) \div {}^{-}3 &= (4 + \tfrac{1}{3}) \cdot {}^{-}\tfrac{1}{3} \\ &= (4 \cdot {}^{-}\tfrac{1}{3}) + (\tfrac{1}{3} \cdot {}^{-}\tfrac{1}{3}) \\ &= {}^{-}\tfrac{4}{3} + {}^{-}\tfrac{1}{9}, \text{ or } {}^{-}(\tfrac{4}{3} + \tfrac{1}{9}), \\ &\qquad \text{or } {}^{-}\tfrac{13}{9}\end{aligned}$$

a. $(2 + \frac{2}{5}) \div {}^-3$

b. $({}^-3 + \frac{8}{9}) \div \frac{3}{2}$

c. $(4 + {}^-\frac{5}{9}) \div {}^-\frac{5}{3}$

d. ${}^-(8 + \frac{1}{2}) \div {}^-4$

e. ${}^-(4 + \frac{3}{7}) \div \frac{2}{3}$

f. $(14 + {}^-\frac{4}{9}) \div {}^-\frac{7}{3}$

2. Divide:

Example:

$$\begin{aligned} \frac{{}^-8 + 12}{{}^-2} &= {}^-\tfrac{1}{2} \cdot ({}^-8 + 12) \\ &= {}^-\tfrac{1}{2} \cdot {}^-8 + {}^-\tfrac{1}{2} \cdot 12 \\ &= 4 + {}^-6 \text{ or } {}^-2 \end{aligned}$$

a. $\dfrac{15 + {}^-3}{{}^-3}$

b. $\dfrac{{}^-18 + {}^-21}{{}^-3}$

c. $\dfrac{{}^-30 + 25}{10}$

d. $\dfrac{14 + {}^-21}{{}^-7}$

e. $\dfrac{\frac{1}{4} + {}^-\frac{1}{6}}{{}^-\frac{1}{2}}$

f. $\dfrac{{}^-\frac{3}{8} + \frac{7}{6}}{\frac{3}{2}}$

MORE ABOUT FRACTIONAL SYMBOLS AND DIVISION

We have used fractional symbols for rational numbers a great deal, but we need to learn more about them.

4.14 Let's Explore

The symbol ‘${}^-\frac{2}{3}$’ names the additive inverse of $\frac{2}{3}$. The symbol $\dfrac{\text{‘}{}^-2\text{’}}{3}$ means the same as ‘${}^-2 \div 3$’. This of course also means the same as ‘${}^-2 \cdot \frac{1}{3}$’. Why?

1. Copy and complete the table on the following page. First study the example below.

	Meaning of fractional symbol	Definition of division	Simplify		Meaning of fractional symbol	Definition of division	Simplify
$\frac{{}^-2}{3}$	${}^-2 \div 3$	${}^-2 \cdot \frac{1}{3}$	$\frac{{}^-2}{3}$	$\frac{2}{{}^-3}$	$2 \div {}^-3$	$2 \cdot \frac{{}^-1}{3}$	$\frac{{}^-2}{3}$

	Meaning of fractional symbol	Definition of division	Simplify		Meaning of fractional symbol	Definition of division	Simplify
$\frac{{}^{-}4}{5}$	${}^{-}4 \div 5$	${}^{-}4 \cdot \frac{1}{5}$	$\frac{{}^{-}4}{5}$	$\frac{4}{{}^{-}5}$	$4 \div {}^{-}5$	$4 \cdot \frac{{}^{-}1}{5}$	$\frac{{}^{-}4}{5}$
$\frac{{}^{-}6}{7}$				$\frac{6}{{}^{-}7}$			
$\frac{{}^{-}3}{5}$				$\frac{3}{{}^{-}5}$			
$\frac{{}^{-}7}{2}$				$\frac{7}{{}^{-}2}$			
$\frac{{}^{-}\frac{3}{4}}{\frac{5}{7}}$				$\frac{\frac{3}{4}}{{}^{-}\frac{5}{7}}$			

2. a. Multiply $\frac{{}^{-}2}{3}$ by $\frac{{}^{-}1}{{}^{-}1}$.

   b. Multiply $\frac{{}^{-}\frac{3}{4}}{\frac{5}{7}}$ by $\frac{{}^{-}1}{{}^{-}1}$.

   c. Suppose x is any number of arithmetic and y is any non-zero number of arithmetic. Multiply $\frac{{}^{-}x}{y}$ by $\frac{{}^{-}1}{{}^{-}1}$.

You should have discovered an important fact about fractional numerals. The numeral '${}^{-}\frac{4}{5}$' means the additive inverse of $\frac{4}{5}$. The numeral '$\frac{{}^{-}4}{5}$' means the result of dividing ${}^{-}4$ by 5. The numeral '$\frac{4}{{}^{-}5}$' means the result of dividing 4 by ${}^{-}5$. But all three numerals name the same number.

*If '**x**' and '**y**' represent any arithmetic numbers, except that **y** ≠ 0, then '${}^{-}\frac{x}{y}$', '$\frac{{}^{-}x}{y}$', and '$\frac{x}{{}^{-}y}$' all represent the same number.*

Sometimes we write '$-\frac{x}{y}$' instead of '$\frac{{}^{-}x}{y}$'. Have you found that it is often hard to tell just where an additive inverse sign is written? When we write fast we may write it too high or too low. Now we know that such a mistake would not change the name of a number.

EXERCISES

1. For each number, write a fractional symbol with a negative denominator:
   - a. ${}^{-}\frac{3}{4}$
   - b. ${}^{-}\frac{7}{2}$
   - c. $-\frac{5}{9}$
   - d. $\dfrac{{}^{-}\frac{3}{4}}{7}$

2. For each number, write a fractional symbol with a negative numerator:
   - a. ${}^{-}\frac{3}{4}$
   - b. $\dfrac{7}{{}^{-}2}$
   - c. $-\frac{5}{9}$
   - d. $\dfrac{\frac{3}{4}}{{}^{-}7}$

3. For each number, write a fractional numeral with positive numerator and denominator.
   - a. ${}^{-}\frac{3}{4}$
   - b. $\dfrac{7}{{}^{-}2}$
   - c. ${}^{-}\frac{5}{9}$
   - d. $\dfrac{\frac{3}{4}}{{}^{-}7}$

Suppose a fractional symbol has both numerator and denominator negative. How can we write other names?

4.15 Let's Explore

1. a. Multiply $\dfrac{{}^{-}2}{{}^{-}3}$ by $\dfrac{{}^{-}1}{{}^{-}1}$.
   - b. Multiply $\dfrac{{}^{-}\frac{3}{4}}{{}^{-}\frac{5}{7}}$ by $\dfrac{{}^{-}1}{{}^{-}1}$.
   - c. Suppose x is any number of arithmetic and y is any non-zero number of arithmetic. Multiply $\dfrac{{}^{-}x}{{}^{-}y}$ by $\dfrac{{}^{-}1}{{}^{-}1}$.
   - d. What have you proved?

2. a. Is it true that $-\dfrac{{}^{-}2}{3} = {}^{-}1 \cdot \dfrac{{}^{-}2}{3}$? Why?
   - b. Multiply $\dfrac{{}^{-}2}{3}$ by $\dfrac{{}^{-}1}{{}^{-}1}$. Is it true that $-\dfrac{{}^{-}2}{3} = -\dfrac{2}{{}^{-}3}$?
   - c. Suppose x is any number of arithmetic and y is any non-zero number of arithmetic. Is it true that $-\dfrac{{}^{-}x}{y} = {}^{-}1 \cdot \dfrac{{}^{-}x}{y}$? Why?

d. Multiply $\frac{{}^{-}x}{y}$ by $\frac{{}^{-}1}{{}^{-}1}$.

e. Is it true that $-\frac{{}^{-}x}{y} = -\frac{x}{{}^{-}y}$?

3. a. Since $-\frac{{}^{-}2}{3} = {}^{-}1 \cdot \frac{{}^{-}2}{3}$, the simplest name for $-\frac{{}^{-}2}{3}$ is ______________ (multiply ${}^{-}1$ by $\frac{{}^{-}2}{3}$).

b. Suppose x is any number of arithmetic and y is any non-zero number of arithmetic. What is the simplest name for $-\frac{{}^{-}x}{y}$?

You have now discovered (and proved) something important about additive inverse signs in a fractional numeral. There are three possible places for a sign to be—one in the symbol above the line, one in the symbol below the line, and one in front.

If there are just two additive inverse signs the numeral names the same number as if there were none. This is true no matter where the two signs are located.

If ‘x’ represents any number of arithmetic and ‘y’ represents any non-zero number of arithmetic, then
‘$\frac{{}^{-}x}{{}^{-}y}$’, ‘$-\frac{{}^{-}x}{y}$’, ‘$-\frac{x}{{}^{-}y}$’ *and* ‘$\frac{x}{y}$’
all represent the same rational number.

We have already proved that if there is just one additive inverse sign in a fractional numeral it doesn't matter where it is located. That is, ‘$\frac{{}^{-}x}{y}$’, ‘$\frac{x}{{}^{-}y}$’ and ‘$-\frac{x}{y}$’ all represent the same number. Now let's investigate the case in which there are three additive inverse signs.

4.16 Let's Explore

1. The symbol ‘$-\frac{{}^{-}2}{{}^{-}3}$’ names the same number as ‘${}^{-}1 \cdot \frac{{}^{-}2}{{}^{-}3}$’. What is the simplest name for $\frac{{}^{-}2}{{}^{-}3}$?

2. Is it true that $-\frac{{}^{-}2}{{}^{-}3} = {}^{-}1 \cdot \frac{2}{3}$? Why?

3. Is it true that $-\frac{{}^{-}2}{{}^{-}3} = -\frac{2}{3}$? Why?

4. Suppose x is any number of arithmetic and y is any non-zero number of arithmetic. Then $-\frac{{}^{-}x}{{}^{-}y} = {}^{-}1 \cdot \frac{{}^{-}x}{{}^{-}y}$.

   What is the simplest symbol for $\frac{{}^{-}x}{{}^{-}y}$?

5. Is it true that $-\frac{{}^{-}x}{{}^{-}y} = {}^{-}1 \cdot \frac{x}{y}$? Why?

6. Is it true that $-\frac{{}^{-}x}{{}^{-}y} = -\frac{x}{y}$? Why?

You have now proved that if there are three additive inverse signs in a fractional numeral it names the same number as if there were just one, regardless of where the one is located.

If 'x' represents any number of arithmetic and 'y' represents any non-zero number of arithmetic, then

'$-\frac{{}^{-}x}{{}^{-}y}$', '$-\frac{x}{y}$', '$\frac{{}^{-}x}{y}$' *and* '$\frac{x}{{}^{-}y}$'

all represent the same number.

Notice that the variables 'x' and 'y' here can never represent negative numbers, according to the above statement. Do you suppose the statement would still be true if we allow negative replacements for 'x' and 'y'? This question will be taken up later, in chapter 6.

EXERCISES

1. For each of these numbers, write 3 different fractional numerals with two additive inverse signs.

Example:

$$\frac{2}{3} = \frac{{}^{-}2}{{}^{-}3} = -\frac{{}^{-}2}{3} = -\frac{2}{{}^{-}3}$$

a. $\frac{3}{4}$ b. $\frac{7}{5}$ c. $\frac{8}{3}$ d. $\frac{12}{5}$

2. For each of these numbers, write 3 fractional numerals using additive inverse signs differently.

Example:

$$-\frac{3}{8} = \frac{{}^{-}3}{8} = \frac{3}{{}^{-}8} = -\frac{{}^{-}3}{{}^{-}8}$$

a. $-\frac{5}{9}$ b. $-\frac{6}{5}$ c. $-\frac{8}{17}$ d. $-\frac{25}{4}$

3. Which of these rational numbers are positive?

a. $\frac{{}^{-}3}{4}$

b. $\frac{{}^{-}7}{{}^{-}6}$

c. $-\frac{{}^{-}6}{5}$

d. $-\frac{4}{{}^{-}3}$

e. $-\frac{{}^{-}3}{{}^{-}5}$

f. $\frac{{}^{-}7}{{}^{-}6}$

g. $\frac{{}^{-}5}{3}$

h. $\frac{47}{{}^{-}17}$

i. $-\frac{{}^{-}35}{12}$

j. $-\frac{75}{{}^{-}31}$

k. $-\frac{{}^{-}17}{{}^{-}12}$

l. $\frac{{}^{-}8}{{}^{-}13}$

4. Which of these sentences are true?

a. $\frac{3}{4} + \frac{6}{{}^{-}7} = \frac{3}{4} - \frac{6}{7}$

b. ${}^{-}7 - \left(-\frac{{}^{-}2}{{}^{-}3}\right) = {}^{-}7 - \left(-\frac{2}{3}\right)$

c. $\frac{4}{5} - \left(-\frac{{}^{-}4}{{}^{-}3}\right) = \frac{4}{5} + \frac{4}{3}$

d. $-\frac{3}{5} + \frac{{}^{-}2}{9} = -\frac{3}{5} - \frac{2}{9}$

e. $8 - \frac{3}{{}^{-}4} = 8 + \frac{3}{4}$

f. ${}^{-}6 + \frac{{}^{-}6}{{}^{-}5} = {}^{-}6 - \frac{6}{5}$

g. $\frac{4}{5} - \frac{{}^{-}2}{{}^{-}3} = \frac{4}{5} + \frac{2}{3}$

h. ${}^{-}6 + \frac{{}^{-}4}{{}^{-}5} = {}^{-}6 + \frac{4}{5}$

AVERAGES

You have found averages before. To find the average of 7 and 5 you first add 7 and 5. Then you divide by 2 (the number of addends). To find the average of 7, 5, and 6 you first add 7, 5, and 6. Then you divide by 3.

We can find averages of rational numbers in the same way. To find the average of $^-6$ and 10, we add $^-6$ and 10. We get 4. Then we divide by 2. The average is 2. Here are some more examples.

Example 1:

Find the average of $^-8$ and 2.

$$\frac{^-8 + 2}{2} = \frac{^-6}{2} = {^-3}$$

Example 2:

Find the average of $^-15$ and 4.

$$\frac{^-15 + 4}{2} = \frac{^-11}{2}$$

Example 3:

Find the average of $^-13$, 6, and $^-5$.

$$\frac{^-13 + 6 + {^-5}}{3} = \frac{^-12}{3} = {^-4}$$

Example 4:

Find the average of $^-2$, 17, $^-29$, and 0.

$$\frac{^-2 + 17 + {^-29} + 0}{4} = \frac{^-14}{4} = \frac{^-7}{2}$$

EXERCISES

Find the average of these rational numbers:

1. 2, 6, $^-3$
2. 4, 9, $^-1$
3. $^-6$, $^-7$, $^-2$, $^-1$
4. $^-4$, 6, $^-5$, 3
5. $\frac{2}{3}$, $^-\frac{3}{6}$, $\frac{4}{5}$
6. $^-\frac{5}{4}$, $^-\frac{7}{8}$, $\frac{1}{2}$
7. 14.6, $^-17.3$, $^-1.35$
8. $^-3.004$, 0.031, 14.042, $^-9.888$

BETWEENNESS AND THE NUMBER LINE

When we studied the number system of arithmetic we talked about betweenness for numbers. Between any two numbers of arithmetic there is another. In fact, there are many numbers between any two numbers of arithmetic. We said the order was *dense*. We proved that there was a number between any two numbers by averaging them. This showed how to find the number halfway between two numbers.

Now let's study betweenness for rational numbers.

4.17 Let's Explore

1. Draw and label a number line for rational numbers.
2. Find the point halfway between:
   - a. 2 and 4
   - b. 2 and $^{-}4$
   - c. $\frac{2}{3}$ and $^{-}\frac{4}{3}$
   - d. 3 and 6
   - e. 3 and $^{-}1$
   - f. $^{-}1.5$ and $^{-}1.6$
   - g. 2 and $^{-}2$
   - h. 2 and $^{-}3$
   - i. $^{-}2.4$ and 0
   - j. $^{-}5$ and $^{-}7$
   - k. $^{-}\frac{1}{2}$ and $^{-}1$
   - l. 1.5 and $^{-}1.6$
   - m. $^{-}3$ and $^{-}6$
3. For each pair of numbers in exercise 2, find the average.

You should have discovered that finding the average of two rational numbers gives the number halfway between them.

We can always find a number between any two rational numbers. This means their order is *dense*.

The order of the rational numbers is ***dense****—between* ***any*** *two rational numbers there is another one.*

We can find the number halfway between two rational numbers by averaging them.

You should remember this about dense order: We say there is one (at least one) number between any two numbers. This automatically means that between two numbers there are many numbers. The number *halfway* between two numbers is a special one. It is easy to find by averaging.

SYMBOLS FOR SETS OF RATIONAL NUMBERS

Suppose we wish to write a name for the set of rational numbers between $^{-}3$ and $\frac{1}{2}$. How many numbers are in this set? You can see that we could not list all of them. We need a way to name such a set—a way in which we don't have to name all the members. That set can be named like this:

$$\{x|\ ^{-}3 < x \text{ and } x < \tfrac{1}{2}\}$$

or

$$\{x|\ ^{-}3 < x < \tfrac{1}{2}\}.$$

The braces show that we are naming a set. The vertical bar separates the symbols in the braces. Before the bar is a variable, 'x'. After the bar we state some rule. That rule tells when a number is in the set. In this case a number is in the set if and only if it is greater than $^{-}3$ and also less than $\frac{1}{2}$ (or *between* $^{-}3$ and $\frac{1}{2}$). We read this symbol like this:

> "The set of all x such that $^{-}3$ is less than x and x is less than $\frac{1}{2}$."

This is a standard way of naming sets when it is difficult or impossible to list all the members.

EXERCISES

1. Write standard set notation for each of these sets. List three members of each set if you can.
   - a. The set of all rational numbers between $\frac{3}{2}$ and 7.
   - b. The set of all rational numbers between $^{-}\frac{5}{2}$ and $^{-}\frac{1}{5}$.
   - c. The set of all rational numbers between $^{-}21.3$ and $^{-}\sqrt{\frac{9}{4}}$.
   - d. The set of all rational numbers between $|^{-}.003|$ and $.003$.
   - e. The set of all rational numbers less than $^{-}1.2$.
   - f. The set of all rational numbers greater than 0.31.
   - g. The set of all rational numbers whose absolute value is less than $\frac{5}{2}$.

2. Plot a graph of each of these sets on a rational number line and describe each graph geometrically.
   - a. $\{y\ |\tfrac{3}{2} < y < 7\}$
   - b. $\{t\ |\ ^{-}\tfrac{5}{2} < t \text{ and } t < {}^{-}\tfrac{1}{5}\}$
   - c. $\{p\ |\ |p| < 2\}$
   - d. $\{p\ |\ |p| \leq 2\}$

e. $\{x \mid {}^-2 \leq x < 2\}$
f. $\{z \mid {}^-\sqrt{\frac{4}{9}} < z \text{ and } z < \sqrt{\frac{16}{25}}\}$
g. $\{y \mid |y| \geq 2\}$

GRAPHS ON A PLANE

We have just plotted graphs on a rational number line. We have also plotted graphs of number sentences for integers, both on a line and on a plane. Remember that for graphs on a plane we need two *axes*—a first axis and a second axis, and we plot ordered pairs. We identify one variable of a sentence with the first axis. Another variable goes with the second axis. These variables are usually taken in alphabetical order.

The graph of the sentence $3 \cdot x = 2 \cdot y$ is a picture of the set of ordered pairs of rational numbers (x, y), for which $3 \cdot x = 2 \cdot y$. More simply, it is a picture of the set

$$\{(x, y) \mid 3 \cdot x = 2 \cdot y\}.$$

The graph on a plane of the sentence $3 \cdot a = 4$ is a picture of the set of ordered pairs (a, b) for which $3 \cdot a = 4$, or of the set

$$\{(a, b) \mid 3 \cdot a = 4\}.$$

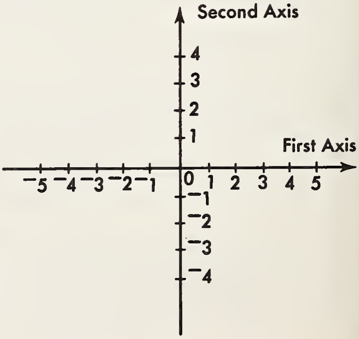
Notice that we always consider *ordered pairs* of numbers for graphs on a plane. If a sentence has only one variable, we must supply another.

We shall now study graphs on a plane for number sentences about rational numbers.

4.18 Let's Explore

1. Let's plot a graph of the sentence $a = b$.

   a. First draw two axes in standard position. Label them "first axis" and "second axis." Select a unit and label integer points on each axis. Do this neatly. Your work should look like the figure at the right.



b. Choose a variable from the sentence $a = b$ for the first axis. Since 'a' comes before 'b' in the alphabet, let's choose 'a' for the first axis.

c. Find some ordered pairs of rational numbers which make $a = b$ true. List the results in a table like this:

a	0	1	2	3	4	$^{-}1$	$^{-}2$	$^{-}3$	$^{-}4$	$\frac{1}{2}$	$^{-}\frac{3}{4}$	etc.
b	0											

d. Plot the graph of each ordered pair listed in your table. Just make a small dot for each one. You need not label the dots.

e. Describe the graph.

2. a. Find some more ordered pairs which make $a = b$ true. Use a table like this:

a	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	1.15	1.25
b											

b. Plot the graphs of these ordered pairs.

c. Describe the graph of $\{(a, b) \mid a = b\}$.

d. How does the graph of $a = b$ for rational numbers differ from the graph of $a = b$ for integers?

e. How far does the graph extend?

3. Plot the graph of $a \geq b$ for rational numbers. It will be a picture of $\{(a, b) \mid a \geq b\}$.

a. Again draw axes and label the first axis 'a'. Label the second axis 'b'.

b. Find some ordered pairs making $a \geq b$ true.

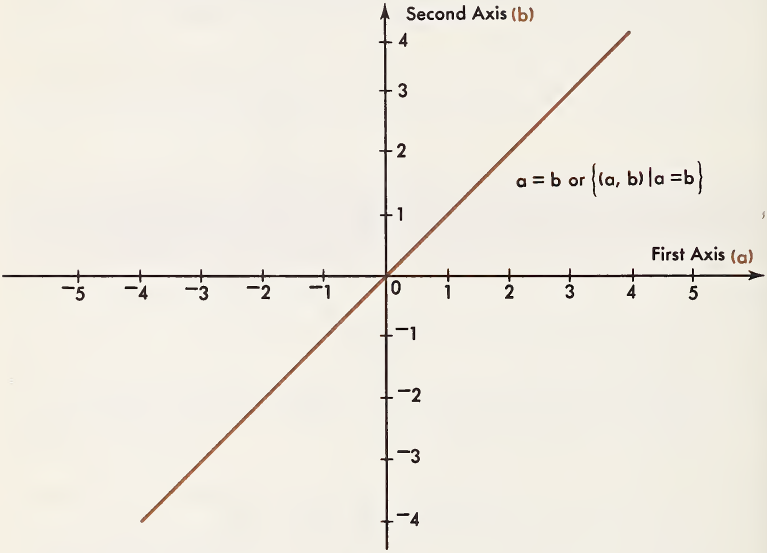
a	3	3	3	3	3	3	$^{-}2$	$^{-}2$	$^{-}2$	etc.
b	3	2	1	0	$^{-}1$	$^{-}5$	$^{-}2$	$^{-}3$	$^{-}5$	

c. Plot the graph of each ordered pair.

d. Describe the graph of $a \geq b$. If necessary, find and plot more points until you can describe the graph.

How did you describe the graph of $a = b$? Did it seem that all the points were on a line? If you said that all the points of the graph are on a line you were correct.

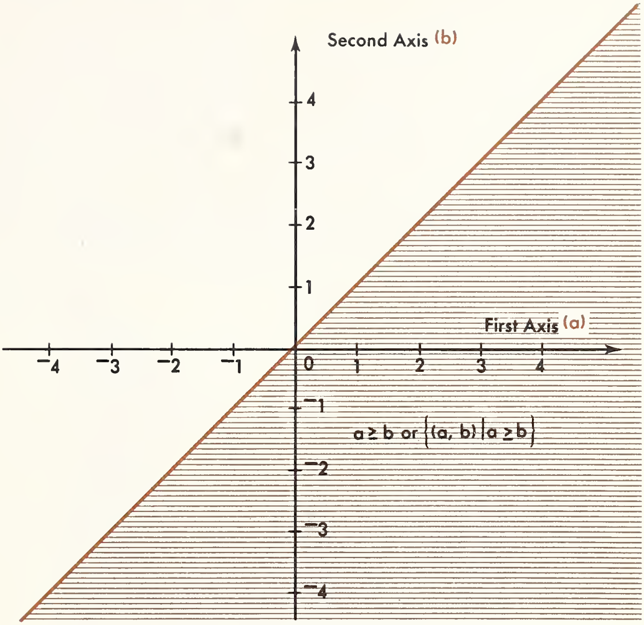
Does the graph of $a = b$ consist of *all* the points on a line? We cannot really tell. There are surely lots of points in the graph. They are very close together. We cannot plot all the points of the graph, even in a very small part of it. So we usually just draw a line for such a graph. Here is the way your graph should look. We could draw arrows on the ends of the graph to show that it goes on and on forever, but usually we do not.



How did you describe the graph of $a \geq b$ (or $\{(a, b) \mid a \geq b\}$)? Did it seem that all the points were on a half-plane? If this is what you said you are incorrect. Remember that a line separates a plane into *three* sets—two half-planes and the line itself. The graph of $a \geq b$ includes the line (the graph of $a = b$). So the graph of $a \geq b$ has all its points either on a half-plane or the line which is its edge.

Does the graph of $a \geq b$ consist of *all* points on the half-plane or the line? We really cannot tell. The points are so close together we cannot plot all of them, even in a small portion of the graph. So we usually draw the line and then shade the half-plane to make the graph.

Here is the way your graph should look.



4.19 Let's Explore

Let's plot the graph of the sentence $a > b$ (or the set $\{(a, b) \mid a > b\}$).

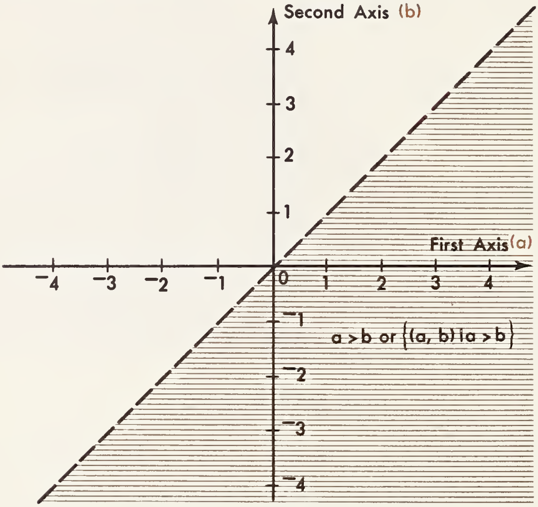
1. Draw axes in standard position and label them 'a' and 'b', as before.
2. Find some ordered pairs making the sentence true.

a	⁻1	⁻1	⁻1	0	2	2	2	2	etc.
b	⁻2	⁻3	⁻4	$^-\frac{1}{2}$	1	$\frac{1}{2}$	0	$^-\frac{1}{2}$	

3. Plot the graph of each ordered pair.
4. Describe the graph of $a > b$. If necessary find and plot more points until you can describe the graph.
5. How does the graph of $a > b$ differ from the graph of $a \geq b$?

How did you describe the graph of $a > b$? Did you notice that it is almost like the graph of $a \geq b$? The only difference is that it has no points on the line for $a = b$. Did you say that the graph of $a > b$ has all of its points on a half-plane? If you did, you were correct.

Does the graph of $a > b$ consist of *all* the points on a half-plane? We cannot really tell. The points are so close together we cannot plot all of them, even in a small portion of the graph. We usually draw the line and then shade the half-plane to make the graph. This time we make the line a dotted line. This shows that no points of the line are in the graph. Here is the way your graph should look.



You will probably find that most graphs you plot in the following exercises will be lines, half-planes, or combinations of lines and half-planes. You should use your imagination as you plot graphs, but you should look for these kinds of sets.

EXERCISES

1. Plot a graph on a plane for each of these sentences for rational numbers and describe the graph geometrically.

   a. $a = 2 \cdot b$
   b. $2 \cdot a = b$
   c. $b = a + 1$
   d. $y = 2 \cdot x + 1$
   e. $y \leq 2 \cdot x + 1$
   f. $y < 2 \cdot x + 1$
   g. $2 \cdot a \geq b$
   h. $2 \cdot a > b$

2. Plot a graph on a plane for each of these sets of ordered pairs of rational numbers:
    a. $\{(x, y) \mid y \geq 2 \cdot x\}$
    b. $\{(a, b) \mid b < a + 1\}$
    c. $\{(x, y) \mid x > {}^{-}\tfrac{2}{3}\}$
    d. $\{(x, y) \mid y = {}^{-}1.5\}$
    e. $\{(a, b) \mid b > 0\}$
    f. $\{(x, y) \mid x \geq y\}$
    g. $\{(x, y) \mid y < {}^{-}3 \cdot x\}$

3. Plot the graphs of each set of ordered pairs and label them '*A*', '*B*', '*C*', etc. Connect the points of each set with line segments, to form polygons. Connect them in the order named and connect the last point named to *A*.

a. A: $(2, 3)$
B: $(5, 1)$
C: $(2, 1)$

b. A: $({}^{-}3, {}^{-}4)$
B: $(2, {}^{-}4)$
C: $(2, 1)$
D: $({}^{-}3, 1)$

c. A: $(0, 0)$
B: $(7, 0)$
C: $(7, 3)$
D: $(0, 3)$

d. A: $({}^{-}4, 1)$
B: $(3, 1)$
C: $(2, 5)$

e. A: $({}^{-}3, {}^{-}4)$
B: $(4, {}^{-}4)$
C: $(5, 2)$
D: $(2, 3)$
E: $({}^{-}3, 2)$

f. A: $({}^{-}3, {}^{-}4)$
B: $(4, {}^{-}4)$
C: $(2, 3)$
D: $({}^{-}3, 2)$
E: $(5, 2)$

g. A: $(0, 0)$
B: $(5, 0)$
C: $(3, 5)$
D: $({}^{-}2, 3)$
E: $({}^{-}5, 5)$
F: $({}^{-}3, {}^{-}4)$

4. For each polygon of exercise 3, tell its type.

5. Which of the polygons of exercise 3 are not simple?

6. Shade the interior of each simple polygon of exercise 3.

7. Draw, in red, the diagonals of the polygons in exercise 3, parts e, f, and g.

8. Find the perimeter of each polygon in exercise 3, parts b and c.

9. Look at exercise 3, part c. Which of these sentences about the figure are true?

a. $\overline{AD} = \overline{BC}$

b. $\overline{AB} \cong \overline{CD}$

c. $\overline{DA} = \overline{AD}$

d. $m\,\overline{AB} = m\,\overline{DC}$

DO YOU KNOW THESE TERMS?

To show that you do know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

multiplicative inverse (163)
reciprocal (163)
additive inverse (164)
additive identity (164)
rational number (164)
positive number (165)
negative number (165)
absolute value (165)
multiplicative identity (176)
exponent (186)
square root (187)
average (197)
dense order (198)
ordered pair (200)
half-plane (202)

CHAPTER SUMMARY

The set of rational numbers (164)
Order and the number line (164)
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Subtraction (169)
Multiplication (170)
A multiplicative identity (176)
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Exponents and square roots (185)
Averages (197)
Betweenness and the number line (198)
Graphs on a plane (200)

REVIEW EXERCISES

1. Name the additive inverse of each of the following numbers:
   a. $\frac{5}{2}$
   b. $^-\frac{3}{4}$
   c. $|^-4|$
   d. 0
   e. $^-2.34$

2. Insert '>', '=', or '<' in order to make each of the following a true sentence:
   a. $^-\frac{2}{3}$ $\quad$ $^-.5$
   b. 3.21 $\quad$ $\frac{23}{7}$
   c. 15% $\quad$ $\frac{3}{20}$
   d. $^-\frac{25}{27}$ $\quad$ $^-\frac{2}{3}$
   e. $\frac{3}{4}$ $\quad$ $^-\frac{3}{4}$

3. Which of these sentences are true?
   - a. $\frac{1}{2} \leq \frac{10}{19}$
   - b. $|{}^-3| < |2|$
   - c. $|{}^-4 + 7| \geq |{}^-7 + 4|$
   - d. $|.35| < |{}^-\frac{3}{8}|$
   - e. $|\frac{5}{6}| > |{}^-\frac{3}{4}|$

4. Write a decimal numeral for each of these rational numbers:
   - a. ${}^-\frac{5}{6}$
   - b. $\frac{13}{7}$
   - c. ${}^-\frac{9}{11}$
   - d. $\frac{25}{4}$
   - e. $\frac{7}{6}$

5. Write the simplest fractional numeral for each of these rational numbers:
   - a. $.3232 \cdots$
   - b. 64%
   - c. ${}^-1.21$
   - d. ${}^-.254254$
   - e. 125%

6. Write the simplest fractional numeral for each of the following:
   - a. $\frac{3}{4} + \frac{5}{6}$
   - b. ${}^-\frac{1}{2} + \frac{3}{2}$
   - c. ${}^-\frac{11}{12} + {}^-\frac{3}{5}$
   - d. $\frac{25}{21} + {}^-\frac{3}{4}$
   - e. ${}^-\frac{5}{7} + {}^-\frac{10}{11}$

7. Write the simplest decimal numeral for each of the following:
   - a. $.34 + .256$
   - b. ${}^-1.5 + {}^-.347$
   - c. ${}^-.547 + .1234$
   - d. $1237.1 + {}^-.12371$
   - e. $25.32 - 11.003$

8. Write the simplest fractional numeral for each of the following:
   - a. $\frac{3}{10} - \frac{1}{3}$
   - b. ${}^-\frac{5}{6} - \frac{5}{7}$
   - c. ${}^-\frac{25}{27} - {}^-\frac{1}{3}$
   - d. $\frac{5}{7} - {}^-\frac{1}{14}$
   - e. $\frac{15}{16} - {}^-\frac{1}{15}$

9. Write the simplest decimal numeral for each of the following:
   - a. $23.1 - 42.12$
   - b. ${}^-11.23 - 11$
   - c. ${}^-2.34 - {}^-1.3$
   - d. $12.34 - {}^-32.41$
   - e. ${}^-15\% - {}^-\frac{1}{2}$

10. Write the simplest fractional numeral for each of the following:
    - a. ${}^-\frac{3}{4} \cdot \frac{1}{2}$
    - b. $\frac{1}{3} \cdot \frac{5}{6}$
    - c. ${}^-\frac{3}{5} \cdot {}^-\frac{5}{3}$
    - d. $\frac{2}{7} \cdot {}^-\frac{7}{15} \cdot \frac{1}{2}$
    - e. ${}^-\frac{9}{10} \cdot {}^-\frac{4}{3} \cdot \frac{4}{3}$
    - f. $\dfrac{{}^-\frac{2}{3}}{\frac{1}{3}}$
    - g. $\dfrac{\frac{5}{6}}{\frac{2}{3}}$
    - h. $\dfrac{{}^-\frac{3}{4}}{{}^-\frac{7}{2}}$
    - i. $\dfrac{\frac{10}{3}}{{}^-\frac{1}{9}}$
    - j. $\dfrac{\frac{3}{4} \cdot {}^-\frac{2}{3}}{{}^-\frac{3}{5} \cdot {}^-\frac{5}{2}}$

11. Write a decimal numeral for each of the following:

a. $(^{-}2)^2$

b. $(^{-}5)^3$

c. $(^{-}4)^3 \cdot (^{-}2)^4$

d. $\dfrac{(^{-}5)^2 \cdot (^{-}2)^2}{(^{-}3)^4}$

e. $\dfrac{(^{-}100)^3}{(^{-}6)^2 \cdot (^{-}3)^4}$

12. Average the following numbers.

a. $\frac{3}{4}, {}^{-}\frac{2}{3}, \frac{1}{2}$

b. $5, {}^{-}6, {}^{-}3$

13. List the properties of the system of rational numbers.

14. What property does the system of rational numbers have that the system of arithmetic numbers does not have?

15. What property does the system of arithmetic numbers have that the system of whole numbers does not have?

16. a. What number systems that you have studied have dense order?
b. What do we mean when we say, "The order is dense"?

17. Plot the graphs on a plane of each of the following sentences:

a. $x = y$

b. $x > y$

c. $x = 2y$

d. $2x = y$

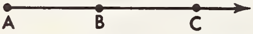
e. $x = y + 1$

f. $x < 2y + 1$

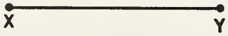
18. Write simplest set notation for each of the following:

a. $\{\cdots {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, \cdots\} \cap \{0, 1, 2, \cdots\}$

b. $\{\cdots {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2 \cdots\} \cup \{1, 2, 3 \cdots\}$

19. 

What is the intersection of $\overset{\circ\!\!\longrightarrow}{BC}$ and $\overline{AB}$?

20. What is the area of $\overline{XY}$? 

21. Mary wished to make some cookies. Her recipe called for the following ingredients:

$\frac{1}{2}$ cup shortening
$\frac{1}{2}$ cup butter
$2\frac{1}{2}$ cups brown sugar
3 eggs
$2\frac{1}{2}$ cups flour
$\frac{1}{4}$ teaspoon salt
$\frac{1}{2}$ teaspoon soda
1 cup chopped pecan meats

Mary found that she only had 2 eggs. If she wished to make the cookies anyway how much of each ingredient should she use?

22. Don delivers the morning newspaper. Twenty of his customers take the paper only on Sundays. Forty of his customers take the paper daily and on Sunday. Five customers take daily papers but not the Sunday paper. The daily papers sell for ten cents each; the Sunday papers sell for twenty cents each; and the rate for both daily and Sunday papers is sixty cents per week. How much does Don make per week if he earns a 20% commission?

23. Mr. Baily owns a square lot which has an area of one acre. How many feet of fencing will be required to put a fence around the lot? Deduct 6 feet for a double gate.

24. A bottling company uses a bottle whose capacity is given by the manufacturer as 1 pt ± $\frac{1}{4}$ oz.
    a. What is the most these bottles could hold when filled to the proper mark?
    b. What is the least these bottles could hold when filled to the proper mark?

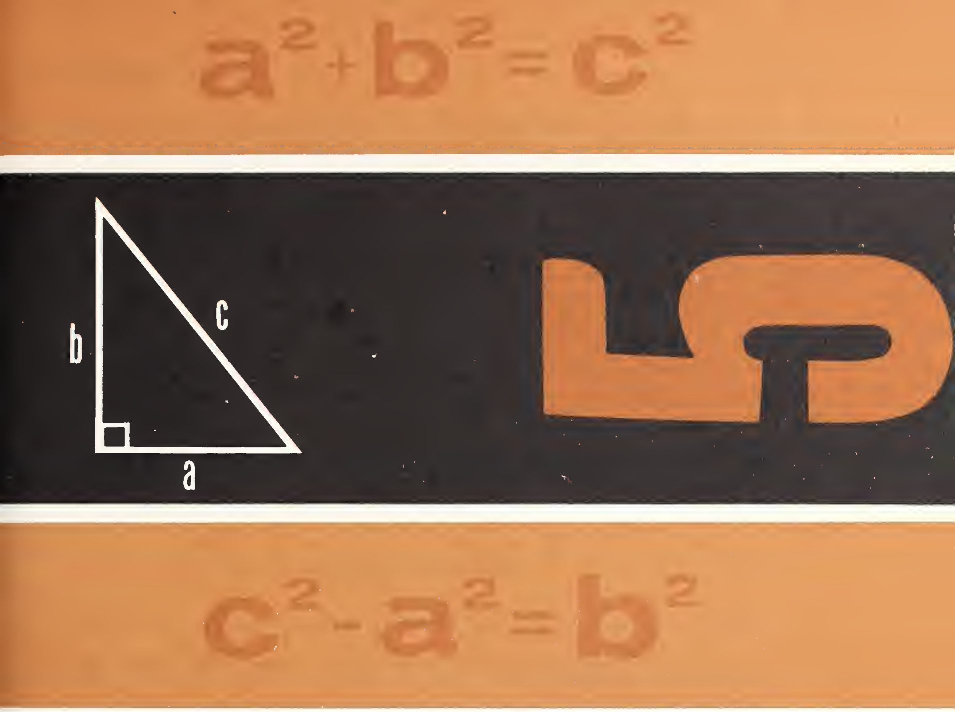
25. Two objects are weighed. x weighs 200 pounds to the nearest pound. y weighs 150 pounds to the nearest half-pound.
    a. Which measure has the greater precision?
    b. Which measure has the smaller relative error?
    c. Which measure has the greater accuracy?

READING LIST

University of Maryland Mathematics Project. *Mathematics for the Junior High School, Second Book.* College Park: Univ of Md., 1959. pp. 13-46.

CHAPTER FIVE

Geometry: Areas, Regular Polygons and Circle Graphs

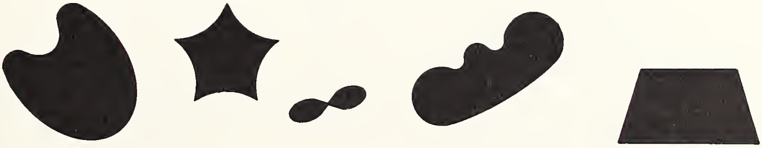


INTRODUCTION

When we studied line segments we developed *measures* for them. Remember that a measure is a number assigned to a line segment. Larger line segments have larger numbers, or measures. The measure of a line segment is also called its *length*. We shall now investigate measures of two-dimensional figures.

MEASURES OF TWO-DIMENSIONAL FIGURES

Do you remember that two-dimensional figures are sets of points like these? They do not have to lie in a plane. But they can always



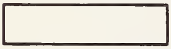
be flattened so that they do lie in a plane. Such figures consist of one or more simple closed curves together with their interiors. Such a set of points is called a *region*. The curve is called the *boundary*, or *edge* of the region.

Regions can be compared in size. You can see that some are larger than others. Can you tell which region shown above is the smallest? Can you order the regions from smallest to largest? By setting up a system of measures, we will have a way of ordering regions by size and also a way of telling how much larger one is than another.

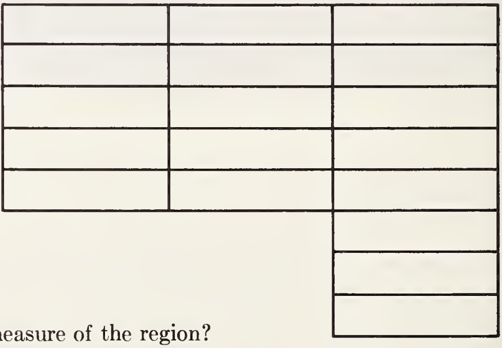
To set up any system of measures, we select an arbitrary member of the set of things we are measuring and assign it the number 1. It is the *unit*. For this kind of measure, then, we shall use a region for a unit, giving it the number one.

5.1 Let's Explore

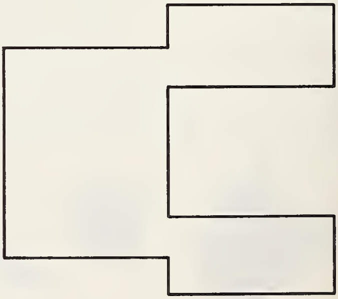
1. Here is a region to be measured. This *rectangular region* has been used as a unit.



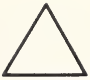
To find the measure of the region, several units have been placed together so that they cover the region. What is the measure of the region?



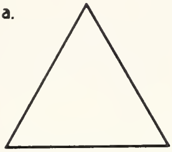
2. Measure the region at the right. Use the rectangular region of exercise 1 as the unit of measure. Copy the rectangular unit of measure of exercise 1 on cardboard and cut it out, so that you can use it.



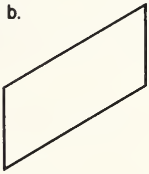
3. Copy this triangular region. Use it as a unit to measure the interior of each polygon.



a.

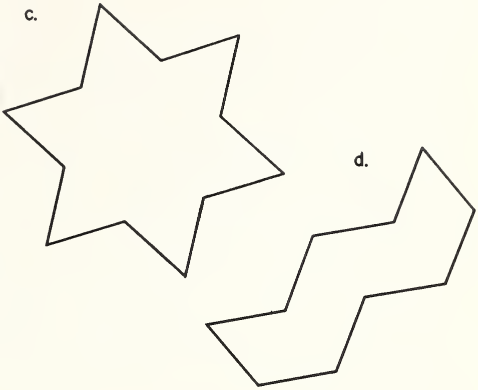


b.

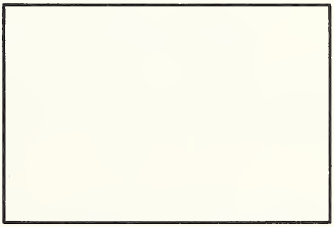


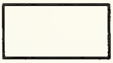
c.

d.

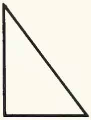


4. Measure this region with each of the following regions as a unit.

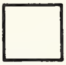




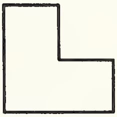
a. rectangular region



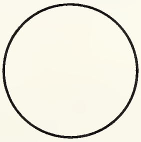
b. triangular region



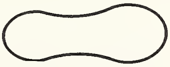
c. square region



e. L-shaped region



d. circular region

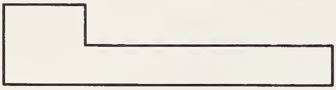


f. simple closed curve region

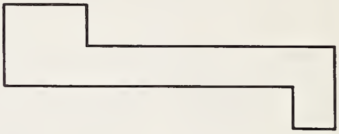
5. In question 4, were any of the unit regions difficult to use? Why?

6. Using the same unit as in question 1, measure these regions.

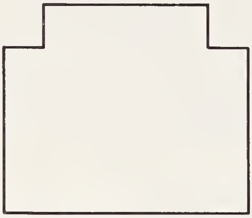
a.



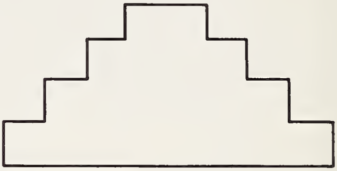
b.



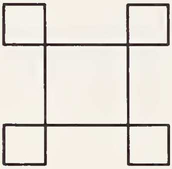
c.



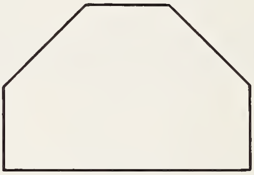
d.



e.



f.



7. What kind of unit region is it easiest to use?

You have probably discovered that to measure a region, unit regions are placed next to each other so that they cover the region to be measured. Any region might be used as a unit, but some of them are not easy to use. The one which is usually used is a square region. The square region is easy to use with most *polygonal regions*, that is, regions whose borders are polygons. It does not work so well with circular regions, but then nothing works nicely with them.

The measure of a one-dimensional set of points is called *length*. We have a word for measures of two-dimensional regions also. They are called *areas*.

The measure of a two-dimensional region is called its ***area.***

The standard units for area are derived from the standard one-dimensional measures. This makes them easier to remember and to use. In the British-American system we use the *square inch*, the *square foot*, the *square yard*, etc. These are regions whose boundary is a square with sides one inch long, one foot long, or one yard long, etc. The relations between these units are given in the following table:

1 square foot = 144 square inches
1 square yard = 9 square feet

Do you see why these statements are true?

The metric system also has standard area units. They are the square centimeter, the square meter, etc. These units are regions whose boundary is a square having a side one centimeter long, one meter long, etc. The table of relations for these units is easy to remember.

1 sq cm = 100 sq mm
1 sq dm = 100 sq cm
1 sq m = 100 sq dm
and so on.

AREA FORMULAS

Mathematicians are always looking for ways to make their work easier. When finding areas, it is often easier to use a formula than to have to actually divide up a region into units. It is possible to develop area formulas for regions determined by many plane figures. You will see how these formulas are developed, how they are used, and how they make it easier to find areas.

RECTANGLES

We will start with the rectangle. Look for an easy way to calculate the area of a rectangular region.

5.2 Let's Explore

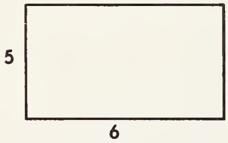
1. Draw a rectangle 5″ long and 3″ wide.
2. Using the square inch as a unit, fill the rectangular region with units.
3. Find its area.
4. How many squares are in each row?
5. How many inches are in the length?

6. Is there one square in a row for each inch in the length?
7. How many rows are in the rectangular region?
8. How many inches are in the width?
9. Is there a row of squares for each inch in the width?
10. Do you see a quick way to calculate the area of a rectangle?

You have probably discovered that a quick way to calculate the area of a rectangular region is to multiply the length by the width. A formula which says this is:

$$A = l \cdot w$$

Here 'A' represents the area, 'l' represents the length, and 'w' represents the width. Using this formula, we see that the area of this rectangular region is 6 × 5, or 30 units. As you can see, the formula is easier to use than dividing the rectangular region into units and counting them.



A square is a special rectangle. Since it is a rectangle, the rectangle formula may be used with it. We can also make up a special formula for finding the area of a square region. Since all sides have the same measure, l and w will always be the same number. Let's use 's' for both the length and the width. Then the formula becomes

$A = l \cdot w = s \cdot s$, where '$s$' represents the measure of the side.
or $A = s^2$

A formula for the area of a rectangular region is $A = l \cdot w$ *where* 'l' *represents length and* 'w' *represents width.*

A formula for the area of a square region is $A = s^2$ *where* 's' *represents the measure of a side.*

You have been reading a symbol like 's^2' as "s exponent 2." Have you heard others read it "s squared"? Perhaps you can now see why this is done. It is because 's^2' represents the area of a square region whose sides have length s.

SEGMENTS

What is the area of a segment? Since a segment does not have any thickness, it seems impossible to divide it up into regions. This is correct.

If we think of a segment as a rectangle and use the formula for the area of a rectangular region, we could say:

$$\begin{aligned} A &= l \cdot w & \\ \text{but}\quad w &= 0. & \text{Why?} \\ \text{Therefore}\quad A &= l \cdot 0 & \\ \text{but } l \cdot 0 &= 0 & \text{Why?} \\ \text{so}\quad A &= 0 & \end{aligned}$$

The area of a segment is zero. This is true for other one-dimensional figures, since they can be straightened out to form segments.

The area of any one-dimensional figure is zero.

Because of this, the area of one-dimensional figures will not be used very much. So we use the phrase "the area of a rectangle" to mean the area of the rectangular region. "The area of a square" will mean the area of a square region, etc. This is shorter and we will understand what is meant.

EXERCISES

1. Fill in the blanks to make a true sentence.

a. 24 in = ____ ft

b. 2640 ft = ____ miles

c. 16 yds = ____ ft

d. 59″ = ____ yds ____ ft ____ in

e. 440 yds = ____ miles

f. 3 yds 5 ft = ____ in

g. 8 ft 2 in = ____ in

h. 340 in = ____ yds ____ ft ____ in

i. 1 mile = ____ in

j. 3 miles 20 yds = ____ ft

k. 63 dm = ____ cm

l. 8 m = ____ cm

m. 5280 m = ____ km

n. 30 dm = ____ m

o. $\frac{1}{2}$ cm = ____ mm

p. $\frac{3}{4}$ m = ____ cm

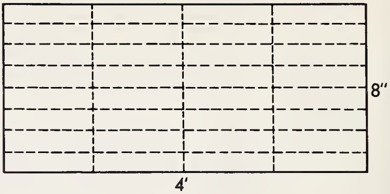
q. 902 m = ____ dm

r. 75 m = ____ km

s. 86 m = ____ dkm ____ m

t. 36 cm 4 dm 9 m = ____ cm

2. Suppose we have a rectangle that is 4 feet by 8 inches. If a rectangle 1 foot by 1 inch is used as a unit, then the region might be divided as in the picture at the right. This unit could be called an inch-foot. What is the area of the rectangle in inch-feet?



We could use the formula for the area of a rectangle.

$$A = l \cdot w$$
$$A = (4\text{ ft}) \cdot (8\text{ in})$$

We can use our experiences with measures with such symbols as '(4 ft) · (8 in)'. We have found that these symbols act very much like numerals. Then

$$\begin{aligned} A &= (4\text{ ft}) \cdot (8\text{ in}) \\ &= 4 \cdot 8\text{ in-ft} \\ &= 32\text{ in-ft} \end{aligned}$$

Now looking at the rectangle we see that this is the area.

Let's find the area of the same rectangle using a square inch as the unit. We already know the width in inches. We shall also need to know the length in inches so that the area will be in square inches.

$$\begin{aligned} l &= 4\text{ ft} = 48\text{ in} \\ w &= 8\text{ in} \end{aligned}$$

Thus,

$$\begin{aligned} A &= l \cdot w \\ &= (48\text{ in}) \cdot (8\text{ in}) \\ &= 48 \cdot 8\text{ in-in} \\ &= 384\text{ in-in} \end{aligned}$$

Here we find the area is 384 inch-inches. But a rectangle that is an inch by an inch is a square, so that 384 inch-inches are 384 square inches. For 384 in-in, or 384 square inches, we may write '384 in^2'. The symbol 'in^2' is a type of exponential notation for 'in · in'.

Suppose we have a rectangle whose area is 32 in-ft and whose length is 4 feet. What is its width? You know

$$A = l \cdot w$$

then $w = \dfrac{A}{l}$ Why?

$$w = \frac{32\text{ ft-in}}{4\text{ ft}}, \text{ or } \frac{32}{4}\,\frac{\text{ft}}{\text{ft}}\,\text{in}$$

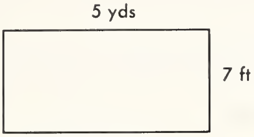
Remember that $\frac{\text{ft}}{\text{ft}}$ acts like an identity for multiplication

So that $w = \frac{32}{4} \cdot 1\text{ in}$

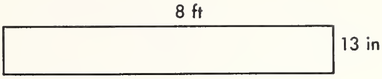
$$w = 8\text{ in}$$

a. Find the area of the following rectangular regions.
Do not change the units.

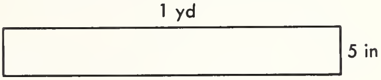
i.



ii.



iii.



b. Show that the width of a rectangle whose area is 36 yard-feet, and whose length is 9 yards, is 4 feet.

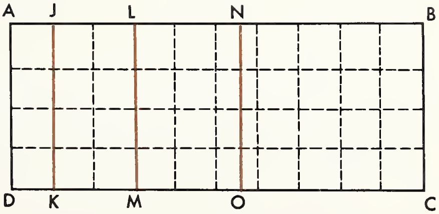
3. A rectangle is 5 yards long and 3 feet wide. What is its area in:
   - a. square feet
   - b. square inches
   - c. yard-feet
   - d. inch-feet
   - e. square yards
   - f. inch-yards

4. An *acre* is a unit of area used to measure large regions. If you divide a square mile into 640 regions with the same measure, each has a measure of one acre. Remember: 640 acres = 1 mile^2.
   - a. How many square feet in 1 mile^2?
   - b. How many square feet in 1 acre?
   - c. A landowner owns a city lot of $\frac{3}{4}$ acre. How many square feet does he own?

SWEEPS

Before we take a look at other regions, it would be helpful if we considered another way of finding formulas for area.

5.3 Let's Explore

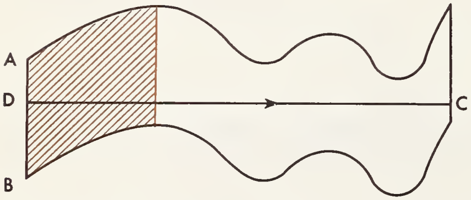


1. Here is a rectangular region $ABCD$, divided into units.

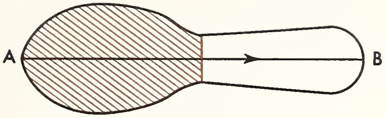
2. Imagine that a segment with the same length as $\overline{AD}$ starts at $\overline{AD}$ and sweeps over all the points of the region. It ends at $\overline{BC}$. While it is sweeping, the segment goes over $\overline{JK}$, $\overline{LM}$, $\overline{NO}$, and all the points of the region.

3. When the segment has gone a distance of DK, how many squares have been swept out?
4. When it has gone a distance of DM, how many squares have been swept out?
5. When it has gone a distance of DO, how many squares have been swept out?
6. What is the total distance the segment goes?
7. How could you find the area if you knew the length of the segment and the distance it sweeps?
8. If we want the number of squares swept, could we use AD, the length of the segment, and some other distance like DC? Why?

You have probably discovered that you can find the area of a rectangle by multiplying the length of a segment that sweeps across the region by the distance it sweeps. The distance must be perpendicular to the segment in order to count the square units. This method of finding area is called *finding area by sweeps*. It can be used in many cases where it would be difficult to count square units. For example, study this region. You can see it would be difficult to divide this region into square units. Suppose the region is of such a type that if we sweep a segment of the proper length across the figure, always keeping it upright, the ends of the segment are always on the top and bottom curves. Then the segment would sweep out all the points of the region. We could then find the area of the region by multiplying the length of the segment by the distance it swept over. In order to count the units, the distance must be measured along a perpendicular to the sweeping segment. So we do not measure the top curve, but instead draw a perpendicular segment to measure. $\overline{DC}$ is such a segment, because it is perpendicular to $\overline{AB}$.



Consider the closed curve at the top of the next page. It defines a region. What is the area of that region? Suppose a segment always perpendicular to $\overline{AB}$ sweeps across the region. If the end-points of the segment are always on the top and bottom curve, the segment will have to change its length as it sweeps. Since the segment changes length, we will have to use the average length of the segment in order to find the area. Sometimes, as in this case, it is hard to see how to find the average length. But in other cases it is not so difficult.

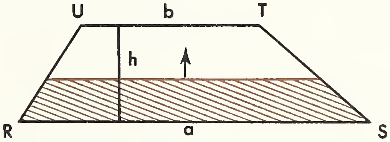


TRAPEZOIDS

We can develop a formula for finding the area of a trapezoid using sweeps.

5.4 Let's Explore

1. Here is a trapezoid $RSTU$, with sides $\overline{RS}$ and $\overline{UT}$ parallel.
   Suppose a segment sweeps across it starting on $\overline{RS}$, with one endpoint always on $\overline{RU}$ and the other end-point always on $\overline{ST}$.
   Does the length of the sweeping segment change?



2. What is the greatest length of the sweeping segment?
3. What is the least length of the sweeping segment?
4. Does the length change uniformly?
5. What is the average length of the sweeping segment?
6. How far does the sweeping segment travel?
7. What is the area of the trapezoid?

You have discovered that the average length of the sweeping segment is $\frac{1}{2} \cdot (a + b)$, and that the distance it sweeps is h. Then the area of the interior is the product $h \cdot \frac{1}{2} \cdot (a + b)$. A formula for the area of a trapezoid is

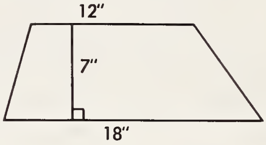
$$A = \tfrac{1}{2} \cdot h \cdot (a + b),$$

where 'a' and 'b' represent the lengths of the parallel sides, and 'h' represents the distance between the parallel sides, called the *height* of the trapezoid. The following example shows how we use this formula.

Example:

Find the area of a trapezoid whose parallel sides are 12 inches long and 18 inches long, if its height is 7 inches.

$$\begin{aligned} A &= \tfrac{1}{2} \cdot h \cdot (a + b) \\ &= \tfrac{1}{2} \cdot 7 \cdot (12 + 18) \\ &= 85 \text{ in}^2 \end{aligned}$$



Review Practice

1. Write fractional numerals for each of these numbers:
   a. 37%
   b. $^-1.25$
   c. $^-.032$
   d. 312%
   e. $^-(1 + \frac{3}{4})$
   f. 10^{-2}
   g. .03%
   h. $^-12.5\%$

2. Write decimal numerals for each of these numbers:
   a. $\frac{60}{5}$
   b. $1\frac{3}{4}$
   c. $\frac{3}{2}$
   d. 47.2%
   e. $^-107\%$
   f. $^-\frac{9}{4}$
   g. .05%
   h. $^-.1\%$

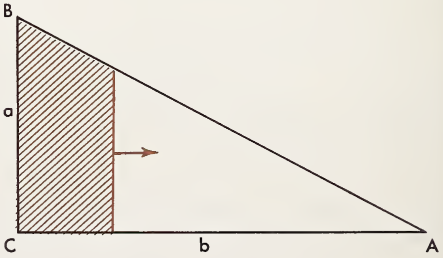
3. Find:
   a. 25% of 12
   b. 200% of $^-17$
   c. $12\frac{1}{2}\%$ of 64
   d. 150% of 3.6
   e. 10% of 3.2×10^2
   f. 1.5% of 300
   g. $37\frac{1}{2}\%$ of 128
   h. $33\frac{1}{3}\%$ of 261

TRIANGLES

A formula for the area of a triangle can be developed easily using sweeps.

5.5 Let's Explore

1. Here is a right triangle with a right angle at C. Suppose a segment sweeps across it, starting on $\overline{BC}$, with one endpoint always on $\overline{BA}$ and the other endpoint always on $\overline{CA}$. Does the length of the sweeping segment change?



2. What is its greatest length?
3. What is its least length?
4. Does the length change uniformly?
5. What is the average length of the segment?
6. What distance does the sweeping segment travel?
7. What is the area of the right triangle?

You have probably discovered that a formula for the area of a right triangle is

$$A = \tfrac{1}{2} \cdot h \cdot b$$

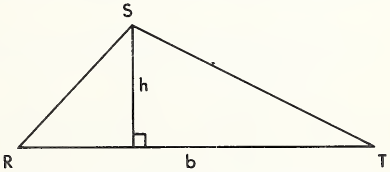
where h and b are the measures of the sides about the right angle. If we take the same triangle and let a segment sweep across the region, starting on $\overline{CA}$, we find that it changes length gradually. Its average length is $\frac{b+0}{2}$, or $\frac{1}{2} \cdot b$. The distance it travels is h. So the area is $(\frac{1}{2} \cdot b) \cdot h$. So a formula for finding the area is:

$$A = \tfrac{1}{2} \cdot b \cdot h$$

Is this the same formula you derived before?

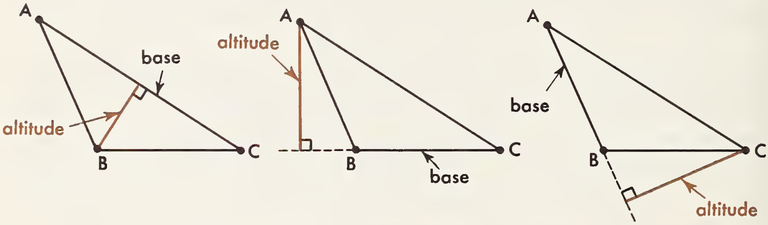
5.6 Let's Explore

1. Now let's consider a triangle which is not a right triangle, $\triangle RST$. Suppose a segment sweeps across the region, starting on $\overline{RT}$. Does the length of the sweeping segment change?



2. What is the greatest length of the sweeping segment?
3. What is the least length of the sweeping segment?
4. What is the average length of the sweeping segment?
5. How far does the segment sweep?
6. What is the area of the triangle?

You have probably discovered that a formula for the area of any triangle is $A = \frac{1}{2} \cdot b \cdot h$, where b is the length of a side of the triangle, and h is the distance from that side to the opposite vertex. The side of the triangle used is called a *base* of the triangle, and the segment perpendicular to the base from the opposite vertex is called an *altitude* of the triangle. Any side of the triangle could be used as a base, and every side has an altitude. The length of an altitude is called a *height* of the triangle. For example, here are three pictures of the same triangle. Study them. In each case, a different side is used as a base. The altitude of a triangle sometimes lies outside the triangle.



When it does, the base has to be extended to meet the altitude.

EXERCISES

1. a. Draw a square whose side is 1′ long.
   b. Show by drawing square inches in it that 1 square foot = 144 square inches.

2. a. Draw a square whose side is 1 decimeter long.
   b. Show by drawing square centimeters that 1 square decimeter = 100 square centimeters.

3. a. Draw a rectangle 8 centimeters by 5 centimeters.
   b. Find its area by drawing square centimeters.
   c. What is its perimeter?
   d. What is the ratio of its length to its width?
   e. What is the ratio of its width to its length?

4. a. Draw a rectangle 5 inches long and 8 centimeters wide.
   b. Using a rectangle region 1 inch by 1 centimeter as a unit, find its area in inch-centimeters.

5. If a rectangle is 20 feet long and 6 inches wide,
   a. What is its area in square feet?, square inches?, inch-feet?

b. What is the ratio of its length to its width?
c. What is the ratio of its width to its length?

6. Find the area of a rectangle whose length and width are as given. Also find the ratio of length to width for each rectangle.

a. $l = 12$ feet
$w = 9$ feet
b. $l = 356$ feet
$w = 188$ feet
c. $l = 1\frac{1}{2}$ centimeters
$w = \frac{3}{8}$ centimeter
d. $l = 8.6$ yards
$w = 13\frac{1}{2}$ feet
e. $l = \frac{1}{2}$ yard
$w = 30$ inches

7. Find the area of a square whose side measures:

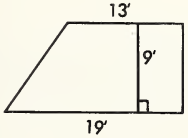
a. 1.4 inches
b. 19 centimeters
c. 9 miles
d. 28.7 feet

8. Find the area of each triangle. Also find the ratio of the height to the length of the base for each triangle.

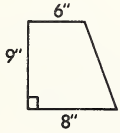
a. $b = 25''$
$h = 14''$
b. $b = 172'$
$h = 100'$
c. $b = 2'$
$h = 28''$
d. $h = 1\frac{1}{7}''$
$b = \frac{22}{5}''$

9. Find the area of these trapezoids:

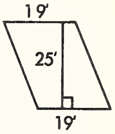
a.



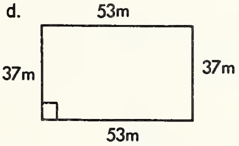
b.



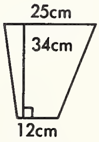
c.



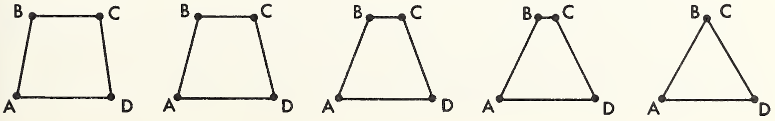
d.



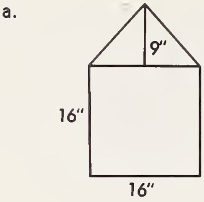
e.

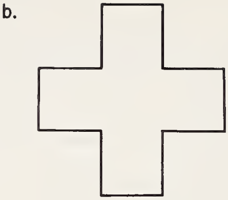


10. These trapezoids suggest that a triangle might be considered as a trapezoid with one side 0 units long. Show that this interpretation of the trapezoid formula $A = \frac{1}{2} \cdot h(a + b)$ gives the triangle formula $A = \frac{1}{2} \cdot b \cdot h$.



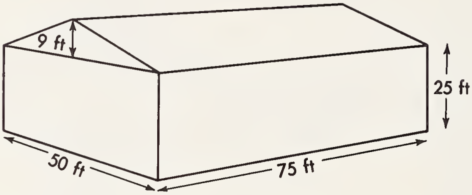
11. Find the area of these figures:





Every side is 3″ long.

12. Find the area of the sides and the ends of the barn pictured here:



13. A room is 13′ by 20′ and the ceiling is 10′ above the floor. In the room there are four windows 5′ by $2\frac{1}{2}$′ and a door $2\frac{1}{2}$′ by 6′. Find the area of the walls and the ceiling.

Let's Estimate

Estimating is useful in many kinds of problems, including finding areas. You should of course always estimate an answer to any problem. Sometimes the estimated result is good enough.

Suppose we want to find the area of a rectangular garden $12\frac{1}{2}$ feet wide and $37\frac{3}{4}$ feet long. We round the numbers and multiply:

$$\begin{aligned} 12\tfrac{1}{2} \times 37\tfrac{3}{4} &\approx 10 \times 40 \\ &\approx 400 \text{ ft}^2 \end{aligned}$$

Estimate the area of each of the following:

1. A rectangular floor measuring 11 feet by $27\frac{1}{4}$ feet.
2. A triangular cloth with a base 27.5″ long and a height of $16\frac{1}{4}$″.
3. A canvas in the shape of a trapezoid with a height of $5\frac{1}{2}$ feet and bases 7 feet long and $9\frac{3}{4}$ feet long.
4. The (rectangular) side of a barn which is 72 feet long and 13 feet high.
5. The triangular end of a chicken house which is 9 feet wide and $6\frac{3}{4}$ feet high.

6. A rectangular region measuring 3.6×10^3 centimeters by 1.5×10^2 centimeters.

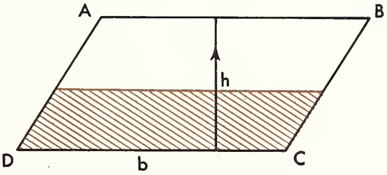
7. A triangular region whose height is 8.21×10^{-3} kilometers and whose base measures 3.02×10^{-2} kilometers.

PARALLELOGRAMS

Can we find a formula for the area of a parallelogram using sweeps?

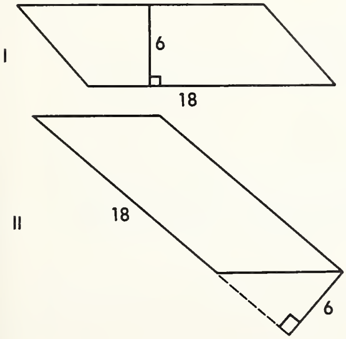
5.7 Let's Explore

1. Here is a parallelogram. Suppose a segment sweeps across the region, starting at $\overline{DC}$, with its endpoints always on $\overline{AD}$ and $\overline{BC}$. Does it change length?



2. What is its length?
3. How far does it sweep?
4. What is a formula for the area of a parallelogram?

You have probably discovered that a formula for the area of a parallelogram is $A = b \cdot h$, where b is the measure of a side of the parallelogram (base) and h is the distance from that side to the opposite side (height). In parallelogram I, the base is 18 units long and the height is 6 units, so the area of the parallelogram is $6 \cdot 18$, or 108 units of area. In parallelogram II, the base is again 18 units long, and the height is 6 units, so the area is $6 \cdot 18$, or 108 units of area.



Do you remember that every parallelogram is a trapezoid? Since this is true, we could also use the trapezoid formula for the area of a parallelogram. Let's try it on the example we just studied. Remember the trapezoid formula

$$A = \tfrac{1}{2} \cdot h \cdot (a + b).$$

In this case $h = 6$, and $a = 18$. Also $b = 18$. Then

$$\begin{aligned} A &= \tfrac{1}{2} \cdot 6 \cdot (18 + 18) \\ &= 6 \cdot \tfrac{1}{2} \cdot (2 \cdot 18) \\ &= 6 \cdot 18 \\ &= 108 \text{ units.} \end{aligned}$$

We could, in fact, derive a formula for the area of a parallelogram from the trapezoid formula. We would simply notice that 'a' and 'b' represent the same number in the case of a parallelogram. Then we would get

$$\begin{aligned} A &= \tfrac{1}{2} \cdot h \cdot (a + b) \\ &= \tfrac{1}{2} \cdot h \cdot (2 \cdot b) \\ &= b \cdot \tfrac{1}{2} \cdot 2 \cdot h \\ &= b \cdot h. \end{aligned}$$

This gives us the same formula we got by sweeping the parallelogram, $A = b \cdot h$.

CONGRUENCE

In your earlier study you found that if two line segments have the same measure then they are congruent. You also found that if two angles have the same measure they are congruent. Perhaps you conjectured that any two figures with the same measure are congruent.

5.8 Let's Explore

1. a. Draw a rectangle 4 centimeters long and 3 centimeters wide.
   b. Draw a rectangle 6 centimeters long and 2 centimeters wide.
   c. Find the area of each rectangle.
   d. Are the rectangles congruent? (Would they fit together exactly?)
2. a. For your first rectangle, what is the ratio of length to width?
   b. For your second rectangle what is the ratio of length to width?

You have seen an example proving that not all figures with the same measure are congruent. Of course if figures are congruent then they have the same measure, but the converse is not true.

Perhaps you have conjectured that if two rectangles have the same area and the ratio of length to width is the same for both then they are congruent. If so, can you prove it?

EXERCISES

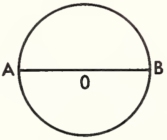
1. Draw two non-congruent triangles with the same area.
2. Draw two non-congruent trapezoids with the same area.
3. Draw two non-congruent one-dimensional figures with the same length.

CIRCLES

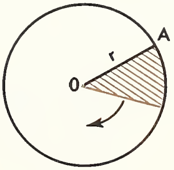
The area of a circle is difficult to find by dividing up the region into units. Can we find a formula for area using sweeps?

5.9 Let's Explore

1. a. Draw a circle with a 2″ radius.
   b. Try to divide the region into square inches.



2. a. Here is a circle with diameter $\overline{AB}$.
   b. Imagine a segment perpendicular to $\overline{AB}$ sweeping across the region.
   c. Does the length of this segment change uniformly?
   d. Does it change faster at the beginning and at the end than it does in the middle, or is the situation reversed?
   e. How can we find the average length of a segment that changes this way?
3. Let's try another way.
   a. Here is a circle with a radius $\overline{OA}$.
   b. Suppose that $\overline{OA}$ sweeps the region by turning like a spoke in a wheel, so that one end stays at O and the other end is always on the circle.



   c. Does the segment change its length?
   d. To find the distance traveled, we have to consider that each point of the segment travels a different distance.
   e. How far does O travel?
   f. How far does A travel?
   g. What is the average of these distances?
   h. Multiply the length of the sweeping segment by the average distance traveled.
   i. What is a formula for the area of the circle?

You have probably discovered that a formula for the area of a circle is

$$A = (\pi \cdot r) \cdot r, \quad \text{or}$$
$$A = \pi \cdot r^2$$

where 'r' represents the measure of the radius.

Did you see that the length of the sweeping segment is always r? Point O traveled zero distance, while point A traveled around the circle, a distance of $2 \cdot \pi \cdot r$. The average distance was $\frac{1}{2} \cdot (2 \cdot \pi \cdot r + 0)$, or $\pi \cdot r$. The product of the length of the segment, r, and the average distance, $\pi \cdot r$, is $\pi \cdot r^2$.

Example:

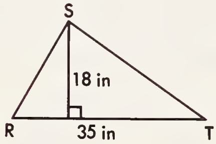
Find the area of a circular region having a radius 3 ft long.

$$\begin{aligned} A &= \pi \cdot r^2 \\ &\approx 3.14 \cdot (3 \text{ ft})^2 \\ &\approx 3.14 \cdot (3 \text{ ft}) \cdot (3 \text{ ft}) \\ &\approx 3.14 \cdot 9 \text{ ft}^2 \\ &\approx 28.26 \text{ ft}^2 \end{aligned}$$

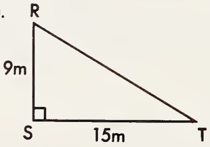
EXERCISES

1. Find the area of each triangle.

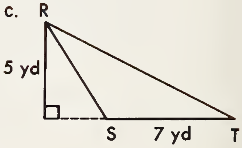
a.



b.

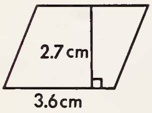


c.

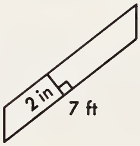


2. Find the area of each parallelogram.

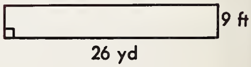
a.



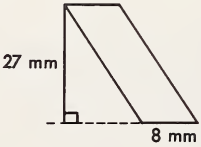
b.



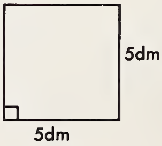
c.



d.

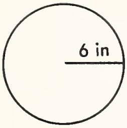


e.

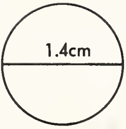


3. Find the area of each circle.

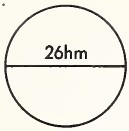
a.



b.



c.



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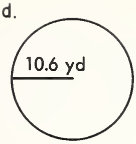
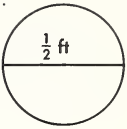
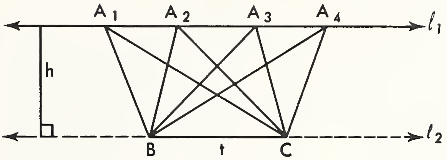


e.



4.



$l_1 \parallel l_2$ All triangles have $\overline{BC}$ as the base. All have a vertex on l_1.

a. What is the area of $\triangle A_1BC$?

b. What is the area of $\triangle A_2BC$?

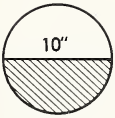
c. What is the area of $\triangle A_3BC$?

d. What is the area of $\triangle A_4BC$?

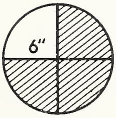
e. Do you have a conjecture about this figure? What is it?

5. Find the area of the shaded regions.

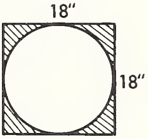
a.



b.



c.



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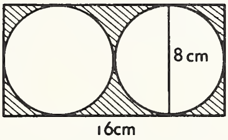
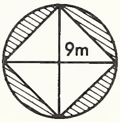
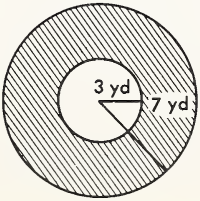


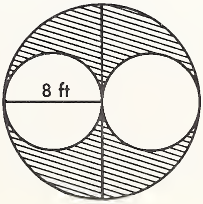
e.



f.



g.

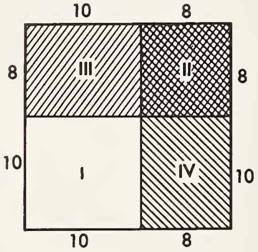


THE PYTHAGOREAN PROPERTY OF RIGHT TRIANGLES

In the days of ancient Greece, a man named Pythagoras found that all right triangles had a certain property. Because he was able to prove this, the property was named after him.

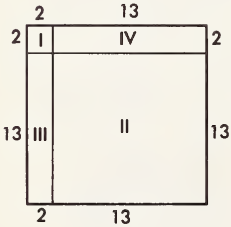
5.10 Let's Explore

1. Study this figure. It is a square region. This region has been divided into four smaller regions, labeled with Roman numerals. The lengths of the different segments are given.
   a. Find the area of each small region.
   b. Find the area of the large square.
   c. Is the area of the large square the same as the sum of the areas of the four regions?

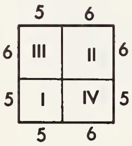


2. Study each of these figures. They are square regions.

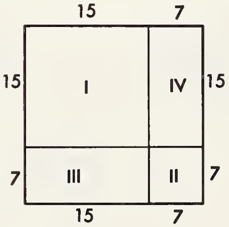
a.



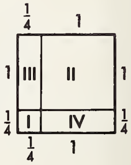
b.



c.



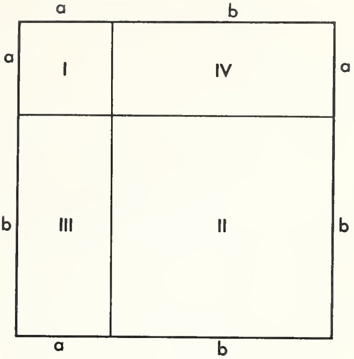
d.



Copy and fill in this chart.

Figure	Area of Region I	Area of Region II	Area of Region III	Area of Region IV	Total Area
a.					
b.					
c.					
d.					

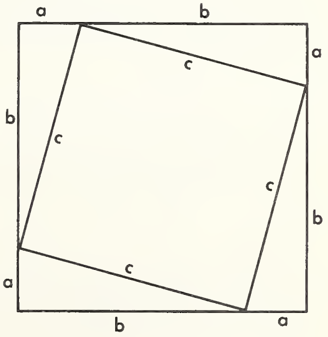
3. Study this figure. It is a square region. The lengths of the segments are not known, but variables indicate what segments have the same length.
   a. Write a number phrase to represent the area of region I.
   b. Write a number phrase to represent the area of region II.
   c. Write a number phrase to represent the area of region III.
   d. Write a number phrase to represent the area of region IV.
   e. Write a number phrase for the sum of these four areas.
   f. Can you simplify it, using the distributive property? Try it.



4. a. The large region is a square whose side measures $(a + b)$. What is the area of the square?
   b. Is $(a + b)^2$ equivalent to your answer to 3-f? Why?
   c. Write a sentence to say that the two phrases are equivalent.

5. You probably have discovered that if the side of a square measures $(a + b)$, then the area of the region, $(a + b)^2$, is also $a^2 + b^2 + 2ab$. Use this fact to find the area of a square whose side measures:
   a. $(3 + 8)$
   b. $(9 + 1)$
   c. $(6 + 4)$
   d. $(2 + 8)$
   e. $(5 + 5)$
   f. $(12 + 7)$
   g. 18
   h. 15

6. a. Draw and cut out a right triangle.
   b. Cut out three more that are just like the first one.
   c. Call the measures of the perpendicular sides a and b. Then each of the four triangles has a side with measure a and a side with measure b. Call the measure of the third sides c.
   d. Draw a square whose sides measure $(a + b)$.
   e. Place the triangles on the square like this.
   f. The interior of the square is divided into five regions. Four of them are triangular. What kind of region do you think the fifth region, the "hole," is?
   g. If you are right, what phrase would represent the area of the hole? Write that phrase.



h. What phrase would represent the area of the other four regions? Write that phrase. (Hint: Put the triangles together so that they form two rectangles.)

i. Write a phrase to represent the area of the large square.

7. a. Now you have seen that the area of the large square region is $c^2 + 2 \cdot a \cdot b$. If you took out the four triangular regions, you would have left a region with what area?

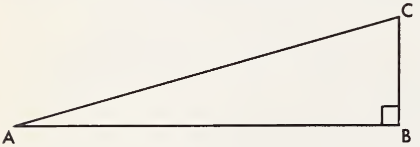
b. Think back to exercise 3. The area of the large square region is also $a^2 + b^2 + 2 \cdot a \cdot b$. If you took out the two rectangular regions, what would be the area of the remaining region?

c. Then how does c^2 compare with $a^2 + b^2$? Write a sentence to say it.

Remember that 'c' represents the length of the longest side of a right triangle. Also 'a' and 'b' represent the lengths of the perpendicular sides. You have discovered that the measures of the sides of a right triangle make this equation true:

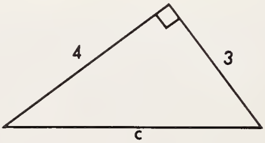
$$a^2 + b^2 = c^2.$$

The longest side, the side opposite the right angle, is called the *hypotenuse*. The other two sides are called *legs*. In the right triangle pictured here, $\overline{AC}$ is the hypotenuse while $\overline{AB}$ and $\overline{BC}$ are the legs.



You do not know if this equation is true for all right triangles because you have not proved the hole is a square. But first let's see if it's worth the trouble. Is this equation useful? Suppose a right triangle has legs which measure 3 and 4. What is the measure of the hypotenuse?

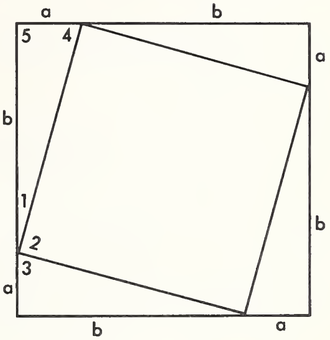
$$\begin{aligned} a^2 + b^2 &= c^2 \\ 3^2 + 4^2 &= c^2 \\ 9 + 16 &= c^2 \\ 25 &= c^2 \end{aligned}$$



What numbers make this equation true? There are two. They are 5 and $^-5$. Both are solutions of the equation. But $^-5$ cannot be a measure for a segment. So only 5 is a solution for the problem.

This equation is useful, since you did not have a way of solving problems like it before. Now let's prove the hole is a square region, so that you will know this equation is true.

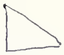
5.11 Let's Explore



The large square was divided into regions like this, and now you want to show that $\angle 2$ is a right angle.

1. What is $m\angle 1 + m\angle 2 + m\angle 3$? Why?
2. Do angles 3 and 4 have the same measure? Why?
3. Remember that $m\angle 5 = 90°$. Then what is $m\angle 1 + m\angle 4$? Why?
4. Look at your answers to 2 and 3. Then answer this: What is $m\angle 1 + m\angle 3$? Why?
5. Look at your answer to exercise 1. Does $(m\angle 1 + m\angle 3) + m\angle 2 = 180°$ say the same thing? Why?
6. Look at your answers to 4 and 5. Then answer this: What is $m\angle 2$?

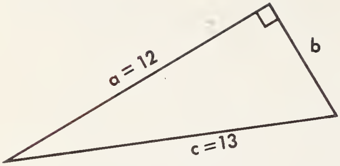
You have now discovered that the hole is really a square region. It is a quadrilateral with four congruent sides. You found that it has one right angle, and you could use the same kind of proof to show that all its angles are right angles. Therefore, the equation $a^2 + b^2 = c^2$ is true about all right triangles. This equation is the Pythagorean property of right triangles.



For any right triangle, $\boldsymbol{a}^2 + \boldsymbol{b}^2 = \boldsymbol{c}^2$ *where* **c** *is the length of the hypotenuse and the legs have lengths* **a** *and* **b**.

You have seen how to use this property to find the length of the hypotenuse. It can also be used if you want to find the length of a leg.

Example:



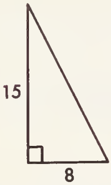
$$\begin{aligned} 13^2 &= 12^2 + b^2 \\ 169 &= 144 + b^2 \\ {}^{-}144 + 169 &= {}^{-}144 + (144 + b^2) \\ 25 &= b^2 \\ b &= 5 \text{ or } b = {}^{-}5 \end{aligned}$$

But $^{-}5$ cannot be the measure of a segment. So, $b = 5$.

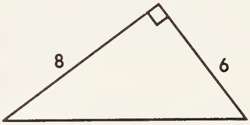
EXERCISES

1. Find the length of the hypotenuse of each of these right triangles.

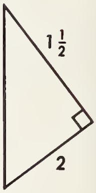
a.



b.

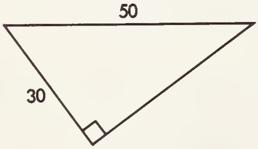


c.

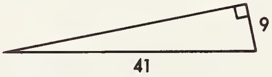


2. Find the length of the leg of each of these right triangles.

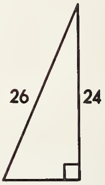
a.



b.

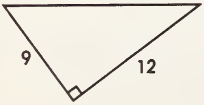


c.

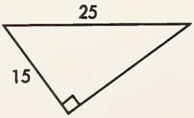


3. Find the missing length in each of these right triangles.

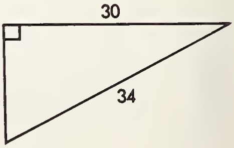
a.



b.



c.



4. How long must a wire be if it is to be fastened on a pole 8′ above the ground and pegged tightly to the ground at a point 6′ from the foot of the pole?

5. What is the length of a diagonal of a rectangle whose sides measure 12 and 5 units?

6. Mr. McGavin goes 9 miles east, 12 miles north and then 7 miles east. How far is he from his starting point?

7. A rectangular field is 80 rods by 150 rods. How much distance do you save if you take a shortcut across the field from one corner to the opposite corner instead of walking around?

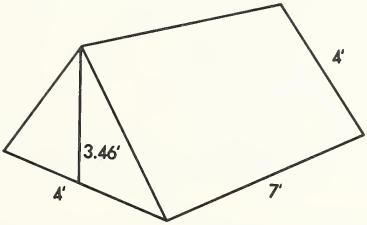
8. What is the largest circular table top that can be brought into a room whose door is 8′ high and 6′ wide?

GEOMETRY AND APPLIED PROBLEMS

Geometric figures are sets of points. Points have no thickness, so they are really *ideas*. We don't find them as objects in the world around us. Geometric figures do represent our ideas of objects in the world, however. When we work applied problems we should think about the appropriate geometric figures.

Example:

Canvas costs $.75 per square foot. How much will the canvas cost to make a tent like this? (The tent does not have a canvas floor. Disregard canvas used in seams.)



To solve this problem we need to know the number of square feet of canvas required. Since each square foot costs $.75 we will then multiply by .75.

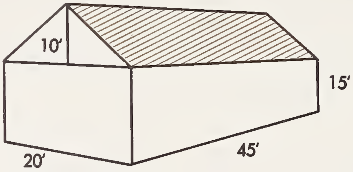
To find the number of square feet we notice that the tent is made up of two rectangles and two triangles. Each rectangle has an area of 28 ft^2. Each triangle has an area of 6.9 ft^2. Altogether $2 \cdot (28 + 6.9)$ ft^2 are needed.

The cost is $.75 × 69.8, or $52.35.

EXERCISES

In the following exercises, don't forget to estimate.

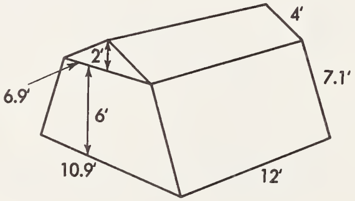
1. Paint costs \$1.37 per gallon and comes only in gallons. One gallon covers about 100 ft^2. How much will the paint cost to give this barn two coats of paint?



2. Mrs. Hodnett has a circular rose garden with a 10 foot diameter. A ten pound bag of fertilizer covers about ten square feet, and costs \$.89. This is the only size bag she can get. How much will it cost her to apply fertilizer to the whole garden?

3. Miss Flagg is buying a wall-to-wall carpet for her living room, which measures 14 feet by 16 feet. The kind of carpet she chose costs \$12.95 per square yard. How much will she pay for the carpet?

4. Canvas costs \$.70 per square foot. How much will the canvas for this tent cost? (The tent does have a canvas floor.)



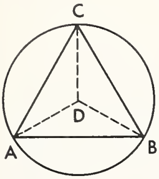
5. To decorate the gym for a party the committee is going to paint 80 old 10″ records and hang them up. A half-pint can of gold paint costs \$.73 and covers about 20 square feet. This is the smallest size can they can buy. How much will it cost to paint all of the records on both sides?

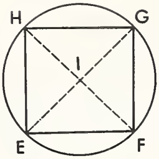
REGULAR POLYGONS AND THE CIRCLE

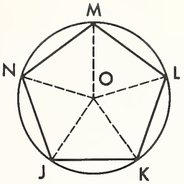
Do you remember the definition of a regular polygon? It is a polygon with all sides congruent and all angles congruent. This uniformity gives a regular polygon a relation to the circle.

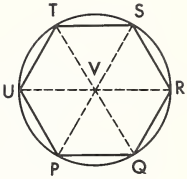
5.12 Let's Explore

1. a. Study these figures. In each case a regular polygon has a circle on its vertices.









In the first figure measure $\angle ADB$, $\angle ADC$ and $\angle BDC$.

b. In each of the other figures measure the *central angles* (the angles with vertex at the center of the circle).
c. What do you discover in each case?
d. What is the sum of $m\ \angle ADC$, $m\ \angle ADB$ and $m\ \angle BDC$?
e. What is the sum of the measures of the central angles in each figure?
f. What is $m\ \angle ADC$?
g. What is $m\ \angle EIH$?
h. What is $m\ \angle NOJ$?
i. What is $m\ \angle UVP$?

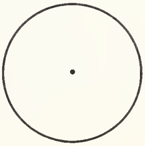
2. Do you now see an easy way to draw a regular polygon? Describe it.

You can probably see that it is possible to have a circle on the vertices of a regular polygon. If segments are drawn from the vertices to the center of the circle, these segments determine central angles which are congruent. You will be able to prove this when you study formal geometry.

The converse relations are also true. They give us a method of constructing regular polygons.

TO CONSTRUCT A REGULAR POLYGON

(For the example let us agree we want a pentagon.)

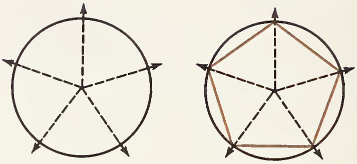


1. Draw a circle and mark its center.
2. Remember, 360 one-degree angles can be placed around the center.

3. Divide the 360° into five angles with the same measure.

$$\begin{array}{r} 72 \\ 5\overline{)360} \\ \underline{35} \\ 10 \\ \underline{10} \end{array}$$

4. Construct five central angles each having a measure of 72°.
5. The intersection of the rays with the circle will be the vertices of a regular pentagon.



A similar procedure will be possible for a regular polygon with any number of sides.

EXERCISES

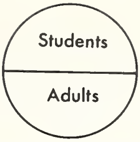
1. Construct a regular triangle.
2. Construct a regular quadrilateral.
3. Construct a regular pentagon.
4. Construct a regular hexagon.
5. Construct a regular heptagon.
6. Construct a regular octagon.
7. Construct a regular nonagon.
8. Construct a regular decagon.

CIRCLE GRAPHS

A circle graph is one type of statistical graph. It consists of a circle, the interior of which represents all, or 100%, of something. This interior is separated into parts by radii of the circle.

Suppose we wished to make a circle graph to show that half of those persons attending the school play were students and the other half were adults. We would draw a circle whose interior represents all of the people attending the play. Then we would separate the interior of the circle into two parts of the same measure.

To do this, we draw a diameter of the circle. (The diameter consists of two radii.) How could we show three parts of the same measure or show parts of different measure?



Attendance at the School Play

5.13 Let's Explore

Refer to the preceding example of a circle graph while answering the following questions.

1. What do we call two rays with a common endpoint?
2. Does each of the radii on the circle graph shown lie on a ray?
3. What is the name of the central angle formed by the rays on which the diameter lies?
4. What is the measure of the angle mentioned in exercise 3?
5. How many angles of this measure are formed by the rays on which the diameter lies?
6. What is the sum of the measures of the angles described in exercise 5?

We call the angle determined by two radii of a circle a *central angle.* Perhaps you already have a pretty good idea about the sum of the measures of the central angles of a circle. Let's investigate further before making a conclusion.

5.14 Let's Explore

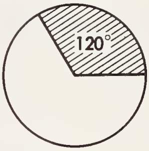
1. Draw a circle.
2. Draw a diameter of your circle.
3. Construct the perpendicular bisector of the diameter.
4. What does perpendicular mean?
5. If two lines are perpendicular what is the measure of the angles determined by their intersection?
6. What is the sum of the measures of the four angles you constructed?

You should now have seen that the sum of the measures of the central angles of a circle is 360°.

If we wished to know what size angle to use to show $\frac{1}{3}$ of a circle we find $\frac{1}{3}$ of 360°.

$$\frac{1}{3} \times 360^\circ = 120^\circ$$

We would then measure a central angle of 120°.

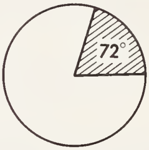


$\frac{1}{3}$ of the interior of a circle

Similarly, if we wish to show 20% of the interior we find 20% of 360°:

$$20\% \times 360^\circ = 72^\circ$$

We could then measure a central angle of 72°.



20% of the interior of a circle

Let us investigate how we would construct a circle graph.

5.15 Let's Explore

We wish to make a circle graph to show the total sales of the week for the Southern Automobile Agency. The agency employs five salesmen. The names of the salesmen and the number of automobiles sold by each are as follows:

Mr. Jones — 3 automobiles
Mr. Kelly — 4 automobiles
Mr. Lee — 6 automobiles
Mr. Self — 8 automobiles
Mr. Huddleston — 9 automobiles

1. What was the total number of sales for the week?

2. What part of the total sales did each man sell?
3. Multiply each of your answers to exercise 2 by 360° to find what size central angles would represent each part.
4. Copy and complete the following chart.

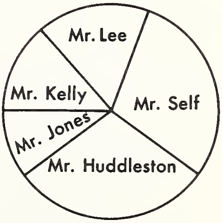
Total Weekly Sales of Southern Automobile Agency

Name of Salesman	No. of Sales	Parts of Total Sales	Measures of Central Angles
Totals			

5. Since all of the interior of the circle is to be used, what must the sum of the measures of the central angles be?
6. a. Draw a circle.
   b. Measure central angles of the sizes indicated in the chart.
7. Be sure to put a title on your graph.

Notice that in the chart you made in exercise 4 you have two opportunities to check to see if your calculations are correct. The total "Parts of Total Sales" should be 1 to represent all, or 100%, of the sales. The total for "Measures of Central Angles" should be 360°. It is important to remember that the part could be named by a fractional numeral, decimal numeral, or a numeral using a per cent sign.

Your graph should look something like this:



Total Weekly Sales of Southern Automobile Agency

EXERCISES

Make circle graphs for each of the following: (you will find it helpful to make a chart for each before attempting the graph).

1. Mary, Jane, and Sally were selling tickets to a play. Mary sold 150 tickets, Jane sold 200, and Sally sold 350. No other tickets were sold.

2. John wished to show on a graph how he spent a typical day. He kept a record of his activities. They were as follows:

   Bathing, Dressing — 1 hr.
   Breakfast — $\frac{1}{2}$ hr.
   School — 6 hrs.
   Lunch — $\frac{1}{2}$ hr.
   Paper Route — $1\frac{1}{2}$ hrs.
   Studying — 2 hrs.
   Dinner — $\frac{1}{2}$ hr.
   Sleeping — 9 hrs.
   Leisure — Remainder

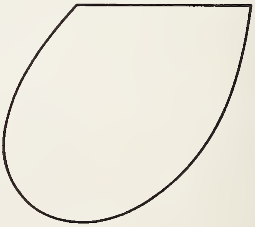
3. The James family had the following expenses for the month of October:

   Food — $200
   House Payment — 125
   Heating — 10
   Electricity — 8
   Dentist — 25
   Recreation — 22
   Water — 5

CONVEX SETS (OPTIONAL)

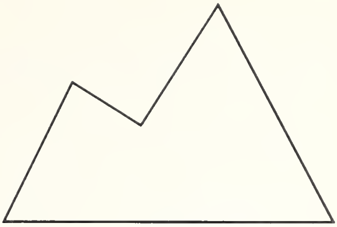
You have studied convex polygons. The interior of a convex polygon is one kind of convex set. There are others.

Let's Explore 5.16



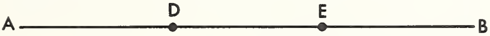
1. a. Here is a region. Copy it.
   b. Take any two points of the region.
   c. Draw the segment joining them.
   d. Is the segment a subset of the region?
   e. Take another pair of points.
   f. Draw the segment joining them.
   g. Is the segment a subset of the region?
   h. Is this true for a segment joining *any* two points of the region?

2. a. Here is another region. Copy it.
   b. Take any two points, and draw the segment joining them.
   c. Is the segment a subset of the region?
   d. Is it possible to find two points for which the segment joining them is not a subset of the region?

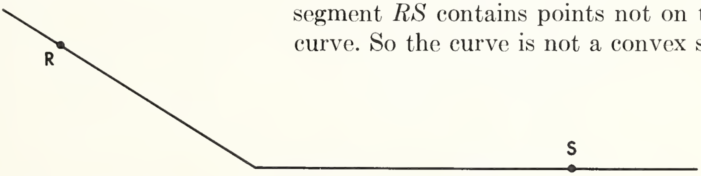


If a region has the property that for *any* two points of the set the segment joining them is a subset of the region, then the region is called a *convex set.*

A segment is a convex set. Here is a segment. If you choose any two points of the segment, such as D and E, $\overline{DE} \subset \overline{AB}$.



Here is another curve. It is not convex. Take points R and S. The segment $\overline{RS}$ contains points not on the curve. So the curve is not a convex set.



We sometimes speak of a curve whose interior is a convex set as a *convex curve.*

EXERCISES

Draw pictures of these sets. Experiment. Decide in each case if the set is convex.

1. a line
2. a rectangular region
3. a half-plane
4. an angle
5. a ray
6. a simple closed curve
7. a triangular region
8. the points inside a simple closed curve
9. a segment
10. an open segment

DO YOU KNOW THESE TERMS?

To show that you know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

measure (221)
length (211)
region (212)
boundary (212)
unit (212)
area (214)
altitude of a triangle (224)
hypotenuse (234)
leg of right triangle (234)
regular polygon (238)
central angle (241)
convex set (245)
convex curve (245)

CHAPTER SUMMARY

Measures of two-dimensional figures (211)
Area Formulas (215)
Rectangles (215)
Segments (217)
Sweeps (219)
Trapezoids (221)
Triangles (222)
Parallelograms (227)
Congruence (228)
Circles (229)
Pythagorean Theorem (232)
Geometry and applied problems (237)
Regular polygon (238)
To construct a regular polygon (239)
Circle graphs (240)
Convex sets (244)

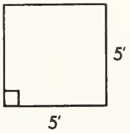
REVIEW EXERCISES

1. What is the name given to a set of points inside a simple closed curve?
2. Is a region two inches square the same size as a region whose measure is two square inches? Explain.
3. What is the area of *any* line segment?
4. Find the area of a rectangle 5′ by 10′.
5. Find the area of a square if each side measures 6 centimeters.

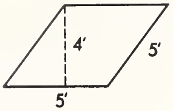
6. Copy and complete the following:
   a. 2 sq ft = ________ sq in
   b. 1 sq yd = ________ sq ft
   c. 5 sq cm = ________ sq mm
   d. 1000 sq dm ________ sq m
   e. 2 sq mi = ________ acres

7. Find the area of the following regions:

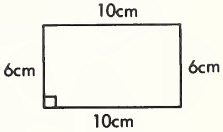
a.



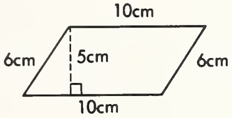
b.



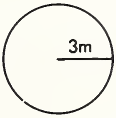
c.



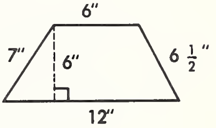
d.



e.



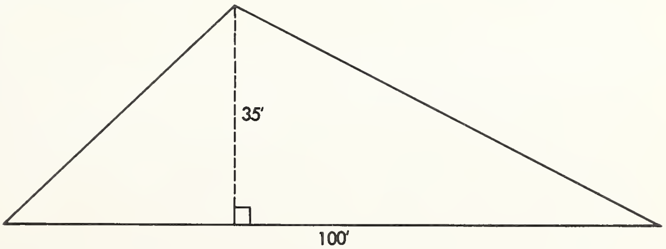
f.



8. Construct each of the following:
   a. an equilateral triangle
   b. a square
   c. a regular hexagon
   d. a regular pentagon
   e. a regular nonagon

9. Mrs. Brown wishes to buy a new carpet for her living room. The living room is rectangular and its measurements are 20′ by 15′. The carpet she wishes to buy costs $10.95 per square yard. Also there is a charge of 20¢ per square foot for binding the edges. Find the total cost of the carpet if she has all four edges bound.

10. Find the cost of seeding the triangular shaped lot sketched below if one box of seed covers 500 square feet and costs $2.98. There is also a 3% sales tax on the seed.



11. If the area of a square plot of ground is 5625 square feet, find the cost of building a sidewalk around two sides of the lot. The cost of building a sidewalk is $1.35 per linear foot.

12. Write scientific notation for two million, three hundred four thousand, fifty five.

13. Write the simplest fractional numeral for each of the following arithmetic numbers:

a. $\dfrac{\frac{3}{4} \cdot \frac{2}{9}}{\frac{5}{8}}$

b. $\dfrac{\frac{7}{5} + \frac{2}{3}}{\frac{3}{4} + \frac{1}{2}}$

c. $\dfrac{3\frac{1}{4}}{2\frac{1}{2}}$

d. $\dfrac{1.75}{2.5}$

e. $\dfrac{\frac{7}{8}}{\frac{1}{2}} + \dfrac{\frac{5}{6}}{\frac{1}{3}}$

14. a. Draw a line segment.
b. Bisect the segment.
c. Bisect one of the angles formed by the segment and its bisector.

15. a. Draw a line segment.
b. Construct another line segment parallel to the one you drew in a.

16. If the measures of two angles of a triangle are 50° and 87°, respectively, what is the measure of the third angle?

17. Write the simplest fractional numeral for each of the following arithmetic numbers;

a. $1.333 \cdots$
b. $.2626 \cdots$
c. $.8666 \cdots$
d. $.4999 \cdots$
e. $124.3737 \cdots$

18. Mary paid $8 for a skirt at a sale. The store advertised that all items had been reduced $33\frac{1}{3}\%$. What was the original price of the skirt?

19. John won the election for class president. The ratio of votes was $\frac{3}{2}$. If there were 1000 votes cast, how many votes did Dan, John's opponent, receive?

20. Find the interest received for one year on $500 invested at 4%, if the interest is compounded semiannually.

21. Write simplest exponential notation for each of the following:

a. $(3^4)^2$

b. $(5^{-2})^5$

c. $\dfrac{1}{(6^{-3})^{-3}}$

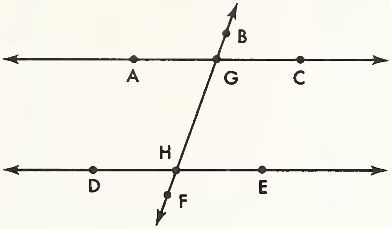
d. $5^7 \cdot 5^0 \cdot 5^{-7} \cdot 2^3$

e. $\frac{1}{2} \cdot 2^5 \cdot \dfrac{1}{2^3} \cdot 2^{-4} \cdot 2^9$

22. Make a circle graph to show the following information:
    Mrs. Eckstein gave a test to her eighth grade class. The grades included 7 A's, 9 B's, 12 C's, 5 D's, and 2 E's.

23. Graph each of the following number sentences:
    a. $x + y = 3$
    b. $x = {}^{-}y$

24. Look at the diagram:



$\overleftrightarrow{AC} \parallel \overleftrightarrow{DE}$ and $m\,\angle AGB = 110°$

a. What is the measure of $\angle HGC$?
b. What is the measure of $\angle BGC$?
c. What is the measure of $\angle DHG$?
d. What is the measure of $\angle FHE$?
e. What is the measure of $\angle AGH$?

25. Which of the following sentences about rational numbers are true?
    a. ${}^{-}|{}^{-}x| = x$
    b. ${}^{-}|a| = {}^{-}a$
    c. $|{}^{-}b| = b$
    d. ${}^{-}|{}^{-}x| + |{}^{-}x| = 0$
    e. $|{}^{-}3| < |3|$

READING LIST

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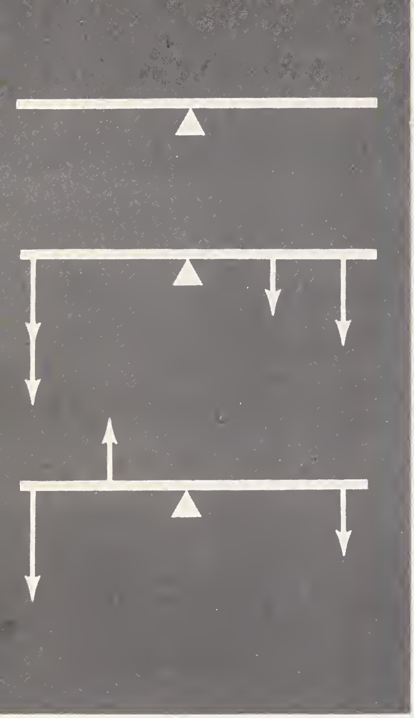


$x + -y = x - y$

6

CHAPTER SIX

More About Integers and Rational Numbers

$$a \cdot (b - c) = a \cdot b - a \cdot c$$

$$2 \cdot n$$

6

$$x \cdot y = 0 \leftrightarrow x = 0 \text{ or } y = 0$$

You have studied the systems of integers and rational numbers and are familiar with their basic properties. There is much more you can learn about these systems.

APPLYING THE SYSTEM OF RATIONAL NUMBERS

The number system of arithmetic was constructed in such a way as to be useful for certain kinds of problems. When we made up the system of integers we did not try especially to make it useful for applied problems. Instead we made it to have certain properties, such as associativity. That system turned out to be useful for applied problems, too, even though it wasn't made for that special purpose. We found it useful with temperature problems, for example.

How was the system of rational numbers constructed? Of course we hoped it would be useful, but we constructed it so that it would have certain properties. This system proved to be useful with exponents as well as temperature problems. We shall find that it is also useful for other kinds of problems.

THE LAW OF THE LEVER

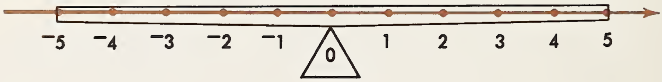
If you have ever been on a see-saw you know something about how to balance one. Suppose you are on one side and a friend is on the other. If your friend weighs more than you do, you know that he will have to sit closer to the center. How much closer?

This is not a question about mathematics alone. Mathematics is about numbers and sets of points, and this question is about a see-saw. A see-saw is an object in the world, and not just an idea, such as we have in pure mathematics.

To get answers to questions like this, we experiment. We cannot find a sure answer by just thinking. When we experiment to find out how the world behaves, we say we are doing *science*. Science is somewhat different from mathematics.

What have scientists found out about see-saws? By experimenting they have learned how see-saws behave, and they can tell your friend where he must sit for a balance. The fact that they have discovered by experimenting is called the *Law of the Lever*. Here is how it works:

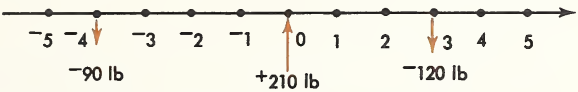
1. Imagine a number line drawn on the see-saw, like this. We usually put the origin at the hinge, or *fulcrum*, but this is not really necessary.



2. Suppose you weigh 90 pounds and your friend weighs 120 pounds. This means that the earth's gravity pulls down on you with a force of 90 pounds. It pulls down on your friend with a force of 120 pounds. There are two forces downward on the see-saw. Here they are represented by arrows.

   Let's suppose you are 4 feet from the hinge, or fulcrum, and your friend is 3 feet from the fulcrum, as shown.

   There are two forces downward on the see-saw totalling 210 pounds. The hinge support must also be pushing up on the see-saw the same amount. If it didn't, the see-saw would fall to the ground. So there is a force upward of 210 pounds.



3. Scientists would agree that we can call upward forces positive and downward forces negative.

   Multiply each force by the rational number that locates it on the line.

   For the $^{-}90$ pound force we get:

   $^{-}4$ ft · $^{-}90$ lb, or ($^{-}4$ · $^{-}90$) ft-lb, or 360 ft-lb

   For the $^{+}210$ pound force we get:

   0 ft · 210 lb, or 0 ft-lb

   For the $^{-}120$ lb force we get:

   3 ft · $^{-}120$ lb, or (3 · $^{-}120$) ft-lb, or $^{-}360$ ft-lb

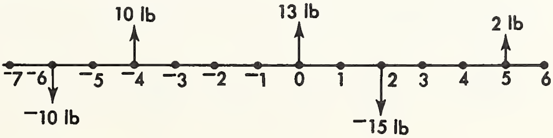
   Scientists call these products *torques* (pronounced *torks*).

   Now add all the torques:

   $360 + 0 + {}^{-}360 = 0$

   The result is zero. By the *law of the lever* the see-saw will be balanced.

The law of the lever works for see-saws. It also works for parts of bridges and machines. Scientists and engineers use it a great deal. We have not yet stated the law of the lever. Before we do, let's take another example, where there are more forces.



Here there are five forces, three positive and two negative. Again let's find the torques. Multiply each force by the rational number that locates it:

$$^{-}6 \text{ ft} \cdot {}^{-}10 \text{ lb},\quad {}^{-}4 \text{ ft} \cdot 10 \text{ lb},\quad 0 \text{ ft} \cdot 13 \text{ lb},$$
$$2 \text{ ft} \cdot {}^{-}15 \text{ lb},\quad \text{and}\quad 5 \text{ ft} \cdot 2 \text{ lb}$$

Now add and simplify. (Since all units are the same, we need not write 'ft-lb' each time.)

$$^{-}6 \cdot {}^{-}10 + {}^{-}4 \cdot 10 + 0 \cdot 13 + 2 \cdot {}^{-}15 + 5 \cdot 2 =$$
$$60 + {}^{-}40 + 0 + {}^{-}30 + 10 = 0$$

The result is zero. According to the law of the lever, everything is in balance.

Now let's state the law of the lever.

When forces act on an object, such as a see-saw, a balance occurs if and only if the sum of the torques is zero.

When you use the law of the lever to see if a see-saw or some other object is balanced, you should first make a sketch. Then draw a number line on it. Then compute the torques and add them. If the sum is zero the object is balanced.

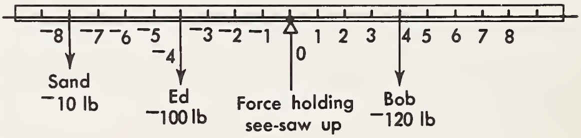
You may place the origin of the number line wherever you wish, but problems are simpler if you put the origin at the fulcrum (when there is one). Then the torque of the force at the fulcrum (the product of the force and the distance from the fulcrum) is zero.

Example:

Bob and Ed are on a see-saw, each 4 feet from the fulcrum. Bob weighs 120 pounds and Ed weighs 100 pounds. Ed has a 10 pound bag of sand, which he places on his end of the board, 8 feet from the fulcrum. Does this balance the board?

To solve this problem:

1. Draw a sketch. Draw a number line on it.



To simplify the problem, we put the origin at the fulcrum.

2. Compute the torques.

   $^{-}8 \cdot {}^{-}10, \quad {}^{-}4 \cdot {}^{-}100, \quad 4 \cdot {}^{-}120$

   (the unit is ft-lb for each)

3. Add the torques. See if the sum is zero.

   $80 + 400 + {}^{-}480 = 0$

   Since the sum of the torques is zero, the law of the lever says that the see-saw will be balanced.

In the preceding example, the force holding the see-saw up would have to be 230 pounds—the sum of the downward forces. We did not need it because its torque is zero. If we had placed the origin somewhere else, then we would have had to use that upward force because its torque would not be zero.

EXERCISES

1. On a see-saw there are several forces. The forces and their distances from a fulcrum are given. Find out whether the see-saw is balanced.
   a. 5 pounds downward, 6 yards to the right.
      2 pounds upward, 2 yards to the right.
      5 pounds upward, 5 yards to the left.
      7 pounds downward, 7 yards to the left.
   b. 3 tons upward, 6 meters to the right.
      2 tons upward, 3 meters to the right.
      15 tons upward, 2 meters to the left.
      2 tons downward, 6 meters to the left.
   c. 1 gram downward, 14 centimeters to the right.
      1 gram downward, 1 centimeter to the left.
      2 grams downward, 2 centimeters to the left.
      3 grams downward, 3 centimeters to the left.
   d. 3 ounces downward, 6 inches to the right.
      2 ounces downward, 1 inch to the right.
      3 ounces upward, 3 inches to the left.
      3 ounces upward, 5 inches to the left.
2. Jerry made a diving board. He laid a board on a rock ledge for a fulcrum. Then he put a 100 pound rock with its center 3 feet from the fulcrum. He found another rock weighing 75 pounds. He put it with its center 1 foot from the fulcrum. (The illustration is given at the top of the next page.)

Jerry weighs 75 pounds. If he walks slowly out on the board 4 feet from the fulcrum, will the board tip?

*3. How far out can Jerry go before the board begins to tip?

MULTIPLICATION OF INTEGERS

When we made up the system of integers we did not define multiplication. In our study of exponents we defined multiplication for integers, but we have not investigated its properties.

Integers are also part of the set of rational numbers. We know the basic properties of multiplication for rational numbers, so we also know them for integers. We know:

Multiplication of integers is an operation (results are unique).
Multiplication of integers is commutative.
Multiplication of integers is associative.
Multiplication of integers is distributive over addition.
The number 1 is the multiplicative identity for integers.

Not every integer has a multiplicative inverse in the system of integers. In fact, there are only two integers which do. What are they?

FACTORING INTEGERS

You have studied factoring of natural numbers. We now wish to study factoring integers. In some ways factoring integers is like factoring natural numbers, and in some ways it is different. How can we factor 6 in the system of integers? Here are four ways:

$$3 \cdot 2, \quad {}^-3 \cdot {}^-2, \quad 1 \cdot 6, \quad {}^-1 \cdot {}^-6$$

How can we factor $^-5$ in the system of integers? Here are two ways:

$$^-1 \cdot 5, \quad 1 \cdot {}^-5$$

The number 6 has eight factors (or divisors) in the system of integers.

They are 1, $^{-}1$, 2, $^{-}2$, 3, $^{-}3$, 6, and $^{-}6$. The number 5 has four factors, 1, $^{-}1$, 5, and $^{-}5$.

6.1 Let's Explore

The integer 12 can be factored in several ways, using at least one negative factor:

$$^{-}1 \cdot {}^{-}12, \quad 2 \cdot {}^{-}2 \cdot {}^{-}3, \quad {}^{-}1 \cdot 2 \cdot {}^{-}6, \quad \text{etc.}$$

1. See how many factorizations with *at least* one negative factor you can find for each of these integers. (Do not use 1 as a factor. Do not use $^{-}1$ more than once in any factorization.)
   a. 16
   b. 24
   c. 32
   d. 36
   e. 100
2. What pattern, or patterns, did you see?
3. See how many factorizations with *at least* one negative factor you can find for each of these integers. (Do not use $^{-}1$ more than once in any factorization.)
   a. $^{-}8$
   b. $^{-}12$
   c. $^{-}36$
   d. $^{-}81$
   e. $^{-}60$
4. What pattern, or patterns, did you see?

You may have discovered an important fact about factorizations. If the number of negative factors is odd, the number named is negative. If the number of negative factors is even, the number named is positive. Is this true if one of the factors is zero?

A factorization of an integer names a negative number when the number of negative factors is odd (*and no factor is zero*). *It names a positive number when the number of negative factors is even* (*and no factor is zero*).

Would the above statement be true if it were made about rational numbers?

EVEN AND ODD INTEGERS

Even numbers were defined to be those numbers having the factor 2. This was in the system of natural numbers. We can define even integers in just the same way.

An integer is even if and only if it has the factor 2.

The odd natural numbers were those which did not have the factor 2. Again, we can make the definition the same way for integers.

An integer is odd if and only if it does not have the factor 2.

EXERCISES

1. Write names for 20 even integers, at least half of them negative.
2. Write names for 20 odd integers, at least half of them negative.
3. Are the even natural numbers also even integers?
4. Are the odd natural numbers also odd integers?
5. Is 0 an odd or even integer, or neither? Why?
6. Is the additive inverse of an even integer also even? Prove your answer.
7. Is the additive inverse of an odd integer also odd? Prove your answer.
8. Is the set of even integers closed under addition?
   Experiment, conjecture, and then see if you can prove or disprove your conjecture. (It may help to recall proofs of similar questions from Book I, Chapter 4.)
9. Is the set of even integers closed under multiplication? Conjecture and prove or disprove your conjecture.

10. Is the set of odd integers closed under addition? Prove your answer.

11. Is the set of odd integers closed under multiplication? Prove your answer.

12. Why do we not define even and odd *rational* numbers?

COMPLETE FACTORIZATION OF INTEGERS

When a natural number is factored completely, each factor named in the factorization is prime. The prime factorization of any natural number is unique. How does this compare with factorization of integers?

6.2 Let's Explore

1. Write as many factorizations as you can for each of these integers. Do not use 1 or $^{-}1$.
   a. $^{-}12$
   b. 12
   c. 150
   d. $^{-}150$

2. Copy and complete this table:

12	$2 \cdot 2 \cdot 3$	$^{-}12$	$^{-}1 \cdot 12$	$^{-}1 \cdot 2 \cdot 2 \cdot 3$
8	2^3	$^{-}8$	$^{-}1 \cdot 8$	$^{-}1 \cdot 2^3$
50		$^{-}50$		
150		$^{-}150$		
30		$^{-}30$		
96		$^{-}96$		
60		$^{-}60$		
90		$^{-}90$		
120		$^{-}120$		

In exercise 2 above you saw a standard kind of factorization for integers. If an integer is positive we factor it into primes as we would in the system of natural numbers. If an integer is negative then we use a factor $^{-}1$; otherwise the factorization is like that for natural numbers. All the other factors are prime natural numbers. If we use only standard factorizations, then every non-zero integer has a unique factorization.

Every non-zero integer has just one standard factorization.

The integer zero has many factorizations. It has no standard factorization. From now on, when we factor integers completely, we shall write their standard factorizations.

EXERCISES

Factor each of these integers completely:

1. 20
2. $^{-}20$
3. $^{-}98$
4. $^{-}68$
5. $^{-}162$
6. $^{-}3150$

Review Practice

1. Simplify:

Example:

$$3^5 \cdot 3^{-2} = 3^3$$

a. $5^4 \cdot 5^2$
b. $4^6 \cdot 4^{-2}$
c. $7^2 \cdot 7^{-5}$
d. $6^{-3} \cdot 6^{-2}$
e. $2^4 \cdot 2^{-5}$
f. $8^4 \cdot 8^{-4}$
g. $3^2 \cdot 3^{-5} \cdot 3^4$
h. $10^4 \cdot 10^{-7} \cdot 10^2$
i. $(^{-}4)^5 \cdot (^{-}4)^2$
j. $(^{-}3)^{-5} \cdot (^{-}3)^7$

2. Simplify:

Example:

$$(5^2)^{-3} = 5^{-6}$$

a. $(6^2)^4$
b. $(5^3)^{-2}$
c. $(4^{-2})^{-4}$
d. $(3^5)^{-1}$
e. $(7^{-5})^2$
f. $(8^{-4})^{-3}$
g. $(6^4)^2$
h. $(6^{-5})^2$

3. Write equivalent symbols without negative exponents:

Examples:

$$3^{-4} = \frac{1}{3^4}$$

$$\frac{4^2}{5^{-3}} = 4^2 \cdot 5^3$$

a. 4^{-1}

b. 3^{-2}

c. $\frac{1}{5^{-3}}$

d. $\frac{1}{4^{-3}}$

e. $3^{-2} \cdot 5^{3}$

f. $6^{-5} \cdot 7^{2}$

g. $\frac{5^{2}}{4^{-3}}$

h. $\frac{8^{3}}{5^{-4}}$

i. $\frac{4^{2} \cdot 5^{-3}}{5^{-2}}$

j. $\frac{3^{5} \cdot 4^{-2}}{6^{-3}}$

k. $\frac{3^{2} \cdot 5^{-3}}{6^{-4} \cdot 7^{3}}$

l. $\frac{4^{-3} \cdot 2^{4}}{3^{-2} \cdot 5^{3}}$

NUMBER PHRASES FOR RATIONAL NUMBERS

We have already studied number phrases, or expressions. Remember, for any number phrase we must think about a replacement set. Number phrases for rational numbers have the set of rational numbers for their replacement set.

Let's recall the definition of a phrase. It is any symbol which either names a number, or *can* name a number for some replacement. We said that two expressions, or phrases, are *equivalent* in certain cases. If they name the same number for *every* replacement, then they are equivalent.

6.3 Let's Explore

1. Let's think about the expression ‘$\frac{x}{x}$’. Make these replacements for ‘x’. In each case, tell what number is named.
   - a. 3
   - b. $\frac{2}{3}$
   - c. $^{-}\frac{1}{2}$
   - d. 6.3
   - e. $^{-}1$
   - f. 0
2. Let's think about this phrase ‘$\frac{x + 1}{x + 1}$’. Make the same replacements as in exercise 1. In each case, tell what number is named.

You have seen that zero is not a sensible replacement in the expression ‘$\frac{x}{x}$’, because ‘$\frac{0}{0}$’ is meaningless. All other rational number replacements are sensible. The expression ‘$\frac{x}{x}$’ names the number 1 for all sensible replacements. In the phrase ‘$\frac{x+1}{x+1}$’ all replacements are sensible except $^-1$. The number 1 is named for all sensible replacements.

Although these phrases do not name the same number for *all* replacements, they do name the same number for all *sensible* replacements. We should then like to call them equivalent. Let us improve our definition of equivalent phrases so that this will be so.

Two number phrases, or expressions, are ***equivalent*** *if and only if they name the same number for all sensible replacements.*

PHRASE SENTENCES

Suppose we wish to say that two phrases are equivalent. We can do it by using the symbol ‘$\equiv$’. Here is a sentence with this verb.

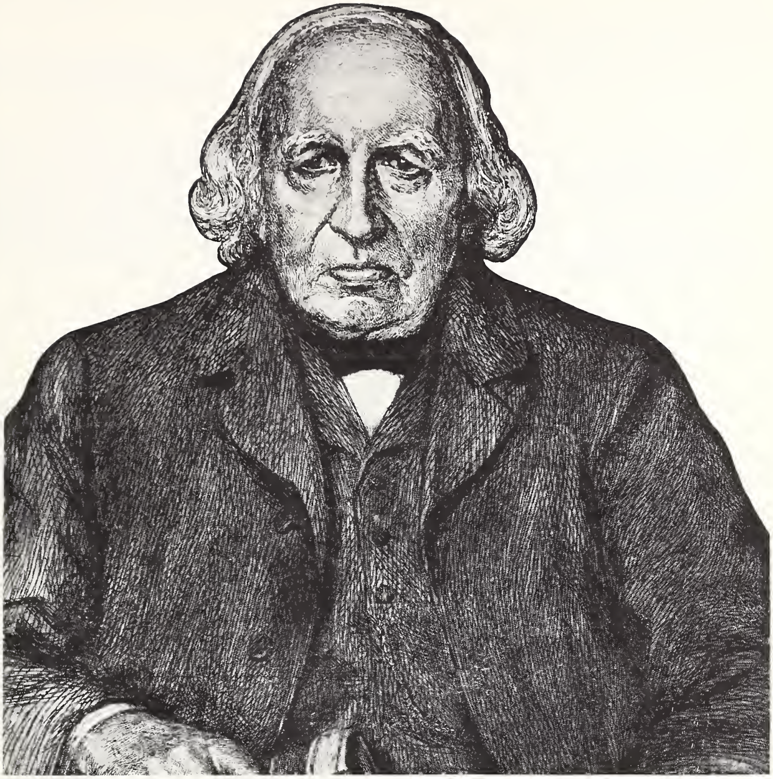
$$\frac{x}{x} \equiv \frac{x+1}{x+1}$$

This sentence says that the two expressions are equivalent. This means that *they will name the same number for all sensible replacements.* We read the symbol ‘$\equiv$’ as “is equivalent to.” Sentences like this one are something like equations. They are not really number sentences, however. This sentence refers to phrases, rather than numbers. So we can call it a “phrase sentence.”

Sometimes phrase sentences with ‘$\equiv$’ for the verb are not called equations but are called *identities*. When we use the word *identity* this way, it has nothing to do with the additive identity nor the multiplicative identity.

PROVING PHRASES EQUIVALENT

In Chapter 1 you worked with pairs of equivalent phrases. You learned about simplifying phrases, and found that it helped in solving equations. In Chapter 4 you learned a little about proving phrases equivalent. For example, you proved that $^-1 \cdot x = {}^-x$, for all rational numbers x. In other words, you proved $^-1 \cdot x \equiv {}^-x$. You also proved



Karl Weierstrass

1815-1897

Weierstrass was a German mathematician who contributed much to advanced mathematics, primarily the calculus. He lived at a time when many properties of this branch of mathematics had been discovered and maybe even proved by different mathematicians. Not all of these mathematicians had the same agreements. He, like Euclid, gathered together these independent studies and organized them so that they were all proved from the same set of agreements. In doing this, he included much of his own work. Weierstrass and Euler make an interesting pair in contrast. Euler had the ability to see many important and subtle relationships in mathematics, but his proofs of these properties were not always acceptable. Weierstrass had the ability to organize properties and their proofs in an acceptable way, but did not do a great deal in finding relationships.

that $^{-}x \cdot y \equiv {}^{-}(x \cdot y)$ and $x \cdot {}^{-}y \equiv {}^{-}(x \cdot y)$. Now we shall learn more about equivalent phrases.

To prove two phrases equivalent, we use the properties of the rational numbers. Here are two phrases we know are equivalent:

$$3 \cdot (y + 4) \quad \text{and} \quad 3 \cdot y + 3 \cdot 4$$

We know they are equivalent because of the distributive property for rational numbers. It tells us that *for any* rational numbers a, b, and c, $a \cdot (b + c) = a \cdot b + a \cdot c$. So no matter what replacements we make for 'y' in the two expressions above, they will name the same number. Then this phrase sentence is true:

$$3 \cdot (y + 4) \equiv 3 \cdot y + 3 \cdot 4$$

Here is another pair of equivalent phrases:

$$x + {}^{-}y \quad \text{and} \quad x - y$$

We know they are equivalent by the definition of subtraction for rational numbers. It tells us that *for any* rational numbers, a and b, $a + {}^{-}b = a - b$. So no matter what replacements we make for 'x' and 'y' in the two phrases, they will name the same number. Then this is a true phrase sentence:

$$x + {}^{-}y \equiv x - y$$

We have already seen that '$^{-}(^{-}x)$' and 'x' are equivalent phrases. At least we studied some examples. How can we *prove* that $^{-}(^{-}x) \equiv x$? We can make several statements about the symbols, proving each statement to be true by some property of rational numbers or by replacing phrases by phrases equivalent to them. Then from the several statements we will know that the original sentence is true. Here is such a proof:

> We know that $^{-}x \equiv {}^{-}1 \cdot x$. We proved it in Chapter 4. This phrase sentence means that whenever we see an additive inverse sign we can replace it by '$^{-}1 \cdot$'. Then the following phrase sentence is true:
>
> $$^{-}(^{-}x) \equiv {}^{-}1 \cdot ({}^{-}1 \cdot x)$$
>
> From this sentence and the associative property of multiplication, we know that
>
> $$^{-}(^{-}x) \equiv ({}^{-}1 \cdot {}^{-}1) \cdot x$$
>
> But $^{-}1 \cdot {}^{-}1 = 1$, and 1 is the multiplicative identity. Therefore the following sentence is true:
>
> $$^{-}(^{-}x) \equiv x$$

We have now proved what we intended. The last sentence says that '$^{-}(^{-}x)$' and 'x' name the same number for all sensible replacements. Are there any non-sensible replacements in this case?

6.4 Let's Explore

1. You have seen that this sentence is probably true:

$$^{-}(a + b) \equiv {}^{-}a + {}^{-}b$$

Let's see if we can prove it.

  a. Remember that $^{-}1 \cdot x \equiv {}^{-}x$. This means that we can replace the additive inverse sign by '$^{-}1 \cdot$' whenever we wish. Do this in the phrase $^{-}(a + b)$.
  b. Use the distributive property to obtain an expression equivalent to your answer to a.
  c. In your answer to b, replace '$^{-}1 \cdot$' by '$^{-}$' both places.
  d. What have you now proved?

2. You have seen that this sentence is probably true:

$$x - y \equiv {}^{-}(y - x)$$

Let's see if we can prove it.

  a. We know that $^{-}(y - x) \equiv {}^{-}(y + {}^{-}x)$. Why?
  b. We also know that $^{-}(y + {}^{-}x) \equiv {}^{-}y + {}^{-}({}^{-}x)$. Why?
  c. We can simplify '$^{-}({}^{-}x)$' to 'x'. Why?
  d. Then from parts b and c we know that

$$^{-}(y + {}^{-}x) \equiv {}^{-}y + \underline{\quad\quad}$$

  e. From parts a and d, we know that $^{-}(y - x) \equiv {}^{-}y + x$. Why?
  f. Use the commutative property to write an expression equivalent to '$^{-}y + x$'.
  g. Use the definition of subtraction to write an expression equivalent to your answer to f.
  h. Use the results of e, f, and g to write the final phrase sentence.

3. See if you can prove $x - y \equiv {}^{-}(y - x)$ by using $^{-}1 \cdot x \equiv {}^{-}x$.

You have just proved pairs of phrases equivalent. We shall find that these results are worth remembering. They tell us something about other phrases, too. Let's look at the phrase '$^{-}(3 \cdot p + q)$'. It looks very much like '$^{-}(a + b)$'. From what we proved, we know that

$$^{-}(3 \cdot p + q) \equiv {}^{-}(3 \cdot p) + {}^{-}q.$$

Look at the expression '$m^2 - 13 \cdot n^3$'. It looks very much like '$x - y$'. From what we have proved, we know that:

$$m^2 - 13 \cdot n^3 \equiv {}^{-}(13 \cdot n^3 - m^2)$$

The facts we have proved about equivalent phrases will be useful and are important. Let's list them and remember them:

$$\begin{aligned} {}^{-}1 \cdot x &\equiv {}^{-}x \\ {}^{-}x \cdot y &\equiv {}^{-}(x \cdot y) \\ x \cdot {}^{-}y &\equiv {}^{-}(x \cdot y) \\ {}^{-}x \cdot y &\equiv x \cdot {}^{-}y \end{aligned}$$

$$\begin{aligned} {}^{-}({}^{-}x) &\equiv x \\ {}^{-}(x + y) &\equiv {}^{-}x + {}^{-}y \\ x - y &\equiv {}^{-}(y - x) \\ {}^{-}x \cdot {}^{-}y &\equiv x \cdot y \end{aligned}$$

You have not yet proved the last of the above sentences, but will be asked to do so in exercise 4 below.

EXERCISES

1. Tell which of these sentences are true.
   - a. ${}^{-}(y + 2) \equiv {}^{-}y + {}^{-}2$
   - b. ${}^{-}p^2 + {}^{-}q \equiv {}^{-}(p^2 + q)$
   - c. ${}^{-}(3 \cdot m - n^2) \equiv n^2 - 3 \cdot m$
   - d. $2 \cdot x^5 - 5 \equiv {}^{-}(2 \cdot x^5 + 5)$
   - e. $3 \cdot y^4 \equiv {}^{-}({}^{-}(3 \cdot y^4))$
   - f. ${}^{-}({}^{-}x^3) \equiv x^3$
   - g. ${}^{-}({}^{-}x)^3 \equiv x^3$
2. For each true sentence of exercise 1, make five different replacements from the set of rational numbers. Show that both expressions name the same number in each case.

Example:

y	${}^{-}(y + 2)$	${}^{-}y + {}^{-}2$
$\frac{1}{2}$	${}^{-}(\frac{1}{2} + 2)$	${}^{-}\frac{1}{2} + {}^{-}2$
	${}^{-}(\frac{1}{2} + 2 \cdot \frac{2}{2})$	${}^{-}\frac{1}{2} + {}^{-}2 \cdot \frac{2}{2}$
	${}^{-}(\frac{1}{2} + \frac{4}{2})$	${}^{-}\frac{1}{2} + {}^{-}\frac{4}{2}$
	${}^{-}\frac{5}{2}$	${}^{-}\frac{5}{2}$

3. For each false sentence of exercise 1, make a replacement for which the two expressions name different numbers.
4. Prove that ${}^{-}x \cdot {}^{-}y \equiv x \cdot y$.
5. In the sentence ${}^{-}1 \cdot a = {}^{-}a$, make these replacements and simplify:

Example:

Replace 'a' by '${}^{-}4$'.

a	${}^{-}1 \cdot a$	${}^{-}a$
${}^{-}4$	${}^{-}1 \cdot {}^{-}4$	${}^{-}({}^{-}4)$
	4	4

a. 2
b. $^{-}3$
c. 0
d. $^{-}\frac{2}{3}$
e. $\frac{4}{3}$
f. $^{-}1.4$

6. In the sentence $^{-}x \cdot y = x \cdot {}^{-}y$, make these replacements and simplify:
a. $5, {}^{-}2$
b. $^{-}13, {}^{-}2$
c. $0, {}^{-}5$
d. $^{-}\frac{7}{9}, 3$
e. $\frac{12}{7}, \frac{4}{5}$
f. $^{-}8.21, {}^{-}6$

7. In the sentence $^{-}x \cdot {}^{-}y = x \cdot y$, make these replacements and simplify:
a. $3, 4$
b. $^{-}\frac{1}{2}, \frac{2}{3}$
c. $^{-}\frac{5}{9}, {}^{-}\frac{7}{3}$
d. $0, 0$
e. $0, {}^{-}4$
f. $0, 1.3$

8. Tell which of these identities are true:
a. $^{-}3 \cdot {}^{-}(2 \cdot x) \equiv 3 \cdot 2 \cdot x$
b. $^{-}1 \cdot (p^2 + 4) \equiv {}^{-}(p^2 + 4)$
c. $^{-}a \cdot {}^{-}(b + 2) \equiv a \cdot (b + 2)$
d. $^{-}(3 \cdot y) \equiv 3 \cdot {}^{-}y$
e. $^{-}x \cdot (4 \cdot y) \equiv 4 \cdot x \cdot y$
f. $(p + q) \cdot {}^{-}t \equiv {}^{-}(p + q) \cdot t$

9. For each true sentence of exercise 8, make five replacements from the set of rational numbers. Show that the two expressions name the same number in each case.

10. For each false sentence of exercise 8, make a replacement for which the two expressions name different numbers.

ANOTHER DISTRIBUTIVE PROPERTY

We know a distributive property for the rational numbers. It is the distributive property of *multiplication* over *addition*. Could multiplication be distributive over another operation? Could addition be distributive over multiplication?

6.5 Let's Explore

1. State the distributive property of multiplication over addition for rational numbers. Do not omit any parentheses.

2. Suppose there were a distributive property of addition over multiplication. How would we state it? (Look at your answer in exercise 1. Interchange plus signs and dots.)

3. Does this property hold in the system of rational numbers? Make some replacements. See if you can make a conjecture and prove it.

4. Suppose there were a distributive property of multiplication over subtraction. How would we state it?
   For any rational numbers a, b, c, $a \cdot (b - c) =$ ________.

5. Does this property hold in the system of rational numbers? Try these replacements. See if you can make a conjecture. Make some more replacements if you wish.

a	b	c	a · (b − c)	(a · b) − (a · c)
2	$\frac{1}{2}$	4	$2 \cdot (\frac{1}{2} - 4)$ or $^-7$	$(2 \cdot \frac{1}{2}) - (2 \cdot 4)$ or $^-7$
$^-2$	$\frac{1}{4}$	$^-4$		
$^-\frac{2}{3}$	$^-6$	$^-3$		
0	$^-4$	$\frac{1}{2}$		
$^-\frac{2}{5}$	0	$^-5$		

You have probably conjectured that the rational numbers do have another distributive property. It is the distributive property of *multiplication* over *subtraction*. You should have also proved that there is no distributive property of addition over multiplication in this system.

For any rational numbers ***a*****,** ***b*****,** *and* ***c*****,**
$$\boldsymbol{a \cdot (b - c) = (a \cdot b) - (a \cdot c)}.$$

We should now try to *prove* that this distributive property really holds in the system of rational numbers.

6.6 Let's Explore

Let's prove the distributive property of multiplication over subtraction.

1. Write this sentence on your paper.
$$a \cdot (b - c) \equiv a \cdot (b + {}^-c)$$
   a. Read it.
   b. Explain what it means.
   c. Is it true? Why?

2. Look at the phrase on the right side of the sentence in exercise 1.
   a. Use the distributive property of multiplication over addition to write a phrase equivalent to it.
   b. Complete this phrase sentence, to make it true.

$$(a \cdot {}^{-}c) \equiv {}^{-}(\qquad)$$

   c. Use your answer to part b. Write a phrase equivalent to your answer to part a.
   d. Use the definition of subtraction. Write a phrase equivalent to your answer to part c.

3. You have proved that several expressions are equivalent to each other. List all of them.

4. From your list in exercise 3, pick out the two phrases we want to describe our new distributive property. Write a phrase sentence which says they are equivalent.

5. Read your sentence of exercise 4.

6. Explain what it means.

7. Are there any replacements for either phrase which are not sensible?

You have now proved that the rational number system has another distributive property. It is the distributive property of multiplication over subtraction. *For any* rational numbers a, b, and c, '$a \cdot (b - c)$' and '$(a \cdot b) - (a \cdot c)$' will name the same number. This is an important property, and we shall use it often.

In phrases like '$(a \cdot b) - (a \cdot c)$' we usually omit the parentheses, and agree that '$a \cdot b - a \cdot c$' means the same thing.

EXERCISES

1. Use the distributive property of multiplication over subtraction to write other names for these numbers.

Example:

$$2 \cdot ({}^{-}3 - \tfrac{1}{2}) = 2 \cdot {}^{-}3 - 2 \cdot \tfrac{1}{2}$$

a. $\frac{2}{3} \cdot ({}^{-}2 - 5)$
b. $\frac{4}{5} \cdot (\frac{4}{3} - {}^{-}2)$
c. ${}^{-}\frac{8}{3} \cdot ({}^{-}4 - \frac{7}{5})$
d. $1.2 \cdot ({}^{-}4.6 - {}^{-}5.3)$
e. ${}^{-}\frac{12}{5} \cdot ({}^{-}7 - {}^{-}6)$
f. ${}^{-}19.1 \cdot (0 - {}^{-}12.2)$
g. $(\frac{4}{5} - \frac{2}{3}) \cdot \frac{1}{2}$
h. $(1.3 - {}^{-}6.5) \cdot {}^{-}5.3$

2. Use the distributive property of multiplication over subtraction to factor:

Example:

$$4 \cdot {}^{-}\tfrac{2}{3} - 4 \cdot \tfrac{3}{5} = 4 \cdot ({}^{-}\tfrac{2}{3} - \tfrac{3}{5})$$

a. $6 \cdot {}^{-}\frac{5}{9} - 6 \cdot \frac{2}{3}$
b. $\frac{3}{4} \cdot \frac{7}{13} - \frac{3}{4} \cdot {}^{-}\frac{5}{9}$
c. ${}^{-}15 \cdot \frac{7}{8} - {}^{-}15 \cdot \frac{3}{4}$
d. ${}^{-}6.14 \cdot \frac{3}{5} - {}^{-}6.14 \cdot {}^{-}\frac{4}{3}$
e. ${}^{-}\frac{8}{9} \cdot {}^{-}\frac{5}{7} - {}^{-}\frac{8}{9} \cdot {}^{-}\frac{3}{5}$
f. $1.02 \cdot {}^{-}4.1 - 1.02 \cdot 3.5$
g. $2 \cdot {}^{-}5 - \frac{3}{2} \cdot {}^{-}5$
h. $\frac{3}{5} \cdot \frac{2}{7} - {}^{-}\frac{4}{9} \cdot \frac{2}{7}$
i. $3 \cdot {}^{-}6 - {}^{-}6 \cdot 5$
j. $\frac{2}{3} \cdot {}^{-}\frac{4}{13} - {}^{-}\frac{15}{8} \cdot \frac{2}{3}$

3. You may have wondered whether taking square root is distributive over addition. If it is, then:

$$\sqrt{x + y} \equiv \sqrt{x} + \sqrt{y}$$

This means that the two expressions name the same number for all sensible replacements. (Negative numbers would not be sensible replacements. Why?)

Experiment and make a conjecture about this. Prove your answer if you can.

Let's Learn a Shortcut

We can use the distributive properties to shorten calculations. They make some problems easy enough to do mentally.

To multiply a number by 11:

By the distributive property of multiplication over addition, for any number n,

$$11 \cdot n = (10 + 1) \cdot n, \quad \text{or} \quad 10 \cdot n + 1 \cdot n.$$

So, to multiply a number by 11 we can multiply by 10 and then add the number. This is easy to do mentally:

Example:

$$11 \cdot 35 = 350 + 35, \quad \text{or} \quad 385$$

To multiply a number by 9:

By the distributive property of multiplication over subtraction, for any number n,

$$9 \cdot n = (10 - 1) \cdot n, \quad \text{or} \quad 10 \cdot n - 1 \cdot n.$$

So, to multiply a number by 9 we can multiply by ten and then subtract the number. This is also easy to do mentally.

Example:

$$9 \cdot 47 = 470 - 47, \quad \text{or} \quad 423$$

Try these:

1. 11×27
2. 9×32
3. 11×16
4. 9×26
5. 11×82
6. 9×56
7. 11×317
8. 9×225
9. 11×512
10. 9×450

Make up some problems of your own, and practice this method.

EQUIVALENT FRACTIONAL EXPRESSIONS

In studying rational numbers in Chapter 4 you learned about fractional expressions. Perhaps you should turn back and review. You proved several things there about fractional numerals. For example, you proved that ‘$\frac{{}^-a}{{}^-b}$’ and ‘$\frac{a}{b}$’ name the same number whenever a is any number of arithmetic and b is any non-zero number of arithmetic. Perhaps you wondered whether the statement would still be true if we allowed the variables to represent any rational numbers (except zero for b, of course).

6.7 Let's Explore

1. Let's experiment with the phrases ‘$\frac{x}{y}$’ and ‘$\frac{{}^-x}{{}^-y}$’ by making some replacements. Copy and complete this table. Extend it if you wish.

x	y	${}^-x$	${}^-y$	$\frac{{}^-x}{{}^-y}$	$\frac{x}{y}$
2	${}^-3$	${}^-2$	3	$\frac{{}^-2}{3}$ or $-\frac{2}{3}$	$\frac{2}{{}^-3}$ or $-\frac{2}{3}$
$-\frac{1}{2}$	$\frac{2}{3}$				
${}^-4$	$-\frac{6}{5}$				
0	${}^-5$				
0	$\frac{2}{3}$				

2. Does it appear that $\frac{x}{y} \equiv \frac{{}^-x}{{}^-y}$? That is, do the two fractional phrases name the same number for all sensible replacements?
3. Which replacements are not sensible?
4. Let's see if we can prove the conjecture.
   a. Write the expression ‘$\frac{x}{y}$’.
   b. Multiply by $\frac{{}^-1}{{}^-1}$.
   c. Remember that ${}^-1 \cdot x \equiv {}^-x$. Replace ‘${}^-1 \cdot$’ by ‘${}^-$’ in two places.
   d. What have you now proved?

You have now proved that ‘$\frac{x}{y}$’ and ‘$\frac{{}^-x}{{}^-y}$’ name the same number for all replacements, except of course that zero cannot be used for ‘y’. In other words, the two expressions name the same number for all sensible replacements, or

$$\frac{x}{y} \equiv \frac{{}^-x}{{}^-y}.$$

6.8 Let's Explore

1. Let's experiment with the phrases ‘$-\frac{x}{y}$’ and ‘$\frac{{}^-x}{y}$’ by making some replacements. Copy and complete this table. Extend it if you wish.

x	y	${}^-x$	$\frac{{}^-x}{y}$	$\frac{x}{y}$	$-\frac{x}{y}$
2	${}^-3$	${}^-2$	$\frac{{}^-2}{{}^-3}$ or $\frac{2}{3}$	$\frac{2}{{}^-3}$ or $\frac{{}^-2}{3}$	$\frac{2}{3}$
${}^-\frac{1}{2}$	$\frac{2}{3}$				
${}^-4$	${}^-\frac{6}{5}$				
0	${}^-5$				
$\frac{2}{3}$	0				

2. Does it appear that $-\frac{x}{y} \equiv \frac{{}^-x}{y}$?

3. Which replacements are not sensible?
4. Let's see if we can prove the conjecture.
   a. Write the expression '$-\frac{x}{y}$'.
   b. Remember that ${}^{-}1 \cdot x \equiv {}^{-}x$. Replace '$-$' by '$\frac{{}^{-}1}{1}$', and multiply.
   c. In your answer to b, replace '${}^{-}1 \cdot$' by '${}^{-}$'.
   d. What have you now proved?

You have now proved that

$$-\frac{x}{y} \equiv \frac{{}^{-}x}{y},$$

which means that the two fractional phrases name the same number for all sensible replacements.

EXERCISES

1. a. Prove the equivalence $-\frac{x}{y} \equiv \frac{x}{{}^{-}y}$.
   b. Which replacements (if any) are not sensible?
   c. Make the following replacements, and show that the two expressions name the same number in each case:

x	y	$-\frac{x}{y}$	$\frac{x}{{}^{-}y}$
3	${}^{-}2$	$-\frac{3}{{}^{-}2}$ or $\frac{3}{2}$	$\frac{3}{{}^{-}({}^{-}2)}$ or $\frac{3}{2}$
${}^{-}4$	5		
$\frac{1}{2}$	${}^{-}\frac{2}{3}$		
${}^{-}\frac{3}{2}$	${}^{-}\frac{1}{3}$		
0	${}^{-}5$		
0	$\frac{3}{4}$		
3	$\frac{5}{2}$		

2. a. Prove the equivalence $-\frac{x}{y} \equiv -\frac{{}^{-}x}{{}^{-}y}$.
   b. Which replacements (if any) are not sensible?
   c. Make the same replacements as in exercise 1 c, and show that the two expressions name the same number in each case.

3. a. Prove the equivalence $-\frac{{}^{-}x}{y} \equiv -\frac{x}{{}^{-}y}$.
   b. Which replacements (if any) are not sensible?
   c. Make the same replacements as in exercise 1 c, and show that the two expressions name the same number in each case.

4. a. Prove the equivalence $-\frac{{}^{-}x}{y} \equiv \frac{x}{y}$.
   b. Which replacements (if any) are not sensible?
   c. Make the same replacements as in exercise 1 c, and show that the two expressions name the same number in each case.

In the above exercises you should have proved some more important equivalences. Let's list and remember them.

$$\frac{x}{y} \equiv \frac{{}^{-}x}{{}^{-}y} \qquad\qquad -\frac{x}{y} \equiv -\frac{{}^{-}x}{{}^{-}y}$$

$$-\frac{x}{y} \equiv \frac{{}^{-}x}{y} \qquad\qquad -\frac{{}^{-}x}{y} \equiv -\frac{x}{{}^{-}y}$$

$$-\frac{x}{y} \equiv \frac{x}{{}^{-}y} \qquad\qquad -\frac{{}^{-}x}{y} \equiv \frac{x}{y}$$

Remember, too, that sentences such as these, like equations, are reversible.

Perhaps you noticed that in fractional phrases like these there are three places for an additive inverse sign. We noticed this also for fractional numerals for rational numbers, in Chapter 4. Did you notice that if there are just two inverse signs, or none, that the phrases are equivalent, no matter where the two are located? If there are three signs or just one, the phrases are equivalent, no matter where the one is located.

The phrases you have just proved equivalent tell us something about other phrases too. For example, the fractional phrase ‘$\frac{{}^{-}(3 \cdot a)}{{}^{-}(6 \cdot b)}$’ looks very much like the phrase ‘$\frac{{}^{-}x}{{}^{-}y}$’. We know the latter is equivalent to ‘$\frac{x}{y}$’, and this tells us also that the first phrase is equivalent to ‘$\frac{3 \cdot a}{6 \cdot b}$’.

EXERCISES

1. Tell which of these sentences are true.

a. $\dfrac{{}^{-}(a + 2 \cdot b)}{{}^{-}x} \equiv \dfrac{a + 2 \cdot b}{x}$

b. $-\dfrac{{}^{-}(y + 2)}{y - 2} \equiv \dfrac{y + 2}{y - 2}$

c. $-\dfrac{(p^2 + q)}{{}^{-}(p + 2)} \equiv \dfrac{p^2 + q}{p + 2}$

d. $-\dfrac{{}^{-}(5 \cdot y^2)}{{}^{-}(x + 4 \cdot y)} \equiv \dfrac{5 \cdot y^2}{x + 4 \cdot y}$

e. $-\dfrac{3 \cdot x + 2}{{}^{-}(2 \cdot x^2)} \equiv \dfrac{3 \cdot x + 2}{2 \cdot x^2}$

f. $\dfrac{{}^{-}({}^{-}x)}{{}^{-}(5 \cdot y)} \equiv \dfrac{{}^{-}x}{5 \cdot y}$

g. $\dfrac{{}^{-}(a + 2 \cdot b)}{{}^{-}({}^{-}b)} \equiv -\dfrac{(a + 2 \cdot b)}{b}$

h. $\dfrac{{}^{-}(x - y)}{a + b} \equiv \dfrac{y - x}{a + b}$

2. a. For each true sentence of exercise 1, show that the two fractional phrases name the same number for two different replacements. You choose the replacements.

b. For each false sentence of exercise 1 show that the two fractional expressions name different numbers for some replacement. You find the replacement.

c. For each of the true sentences of exercise 1, what are the non-sensible replacements?

ANOTHER IMPORTANT PROPERTY

When we multiply any number by zero we get zero. Now let's think of a product of two rational numbers, $a \cdot b$. If $a = 0$, then the product will be zero. If $b = 0$, then the product will be zero. If $a = 0$ and $b = 0$, then the product will be zero. We can say all of this as follows:

For any rational numbers a and b, if $a = 0$ or $b = 0$, then $a \cdot b = 0$.

This is a conditional sentence. Let's use an arrow to write it.

For any rational numbers $\boldsymbol{a}$ *and* $\boldsymbol{b}$, $\boldsymbol{a} = 0$ *or* $\boldsymbol{b} = 0 \rightarrow \boldsymbol{a} \cdot \boldsymbol{b} = 0$.

Notice how we used the word 'or' in the antecedent. The antecedent says $a = 0$ or $b = 0$. It will be true if either part is true. It will be true when both parts are true. This is the way we use the word 'or' in mathematics.

The above property is an important one for rational numbers. Not all number systems have it. In the arithmetic of a four-minute clock, $2 \cdot 2 = 0$, yet neither of the factors named in '$2 \cdot 2$' is zero.

Is the converse of the conditional sentence true?

6.9 Let's Explore

1. Write the converse of $a = 0$ or $b = 0 \rightarrow a \cdot b = 0$.
2. If you were to try proving the converse false, what would you need to do?
3. Try some replacements. See if you can prove the converse false.

Example 1: Replace 'a' by '2' and 'b' by '$^-2$'.

$2 \cdot {}^-2 = 0 \rightarrow 2 = 0$ or ${}^-2 = 0$.

The antecedent is false. The consequent is false. (The consequent is false because *both* parts are false. It would be true if one or both parts were true.) Since the antecedent is false the conditional sentence is true.

Example 2: Replace 'a' by '$^-\frac{2}{3}$' and 'b' by '0'.

${}^-\frac{2}{3} \cdot 0 = 0 \rightarrow {}^-\frac{2}{3} = 0$ or $0 = 0$

The antecedent is true. The consequent is true (because one part is true). The conditional sentence is true.

4. Can you find any replacement making the antecedent true and the consequent false?

Perhaps you are convinced that the converse is true. You could not find an example to prove it false, but you did not *prove* it to be true. Perhaps you can prove it later, after you know more about equations. The converse is actually true, so we know that:

*For any rational numbers **a** and **b**, $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \rightarrow \boldsymbol{a} = 0$ or $\boldsymbol{b} = 0$.*

We can combine our two conditional sentences by using a double arrow, like this:

For any rational numbers $\boldsymbol{a}$ *and* $\boldsymbol{b}$, $\boldsymbol{a} \cdot \boldsymbol{b} = 0 \leftrightarrow \boldsymbol{a} = 0$ *or* $\boldsymbol{b} = 0$.

This is a short statement of our important property. We could also state it this way: "If any product is zero, then at least one of the factors is zero; also if any factor of a product is zero, then the product is zero."

Remember this property. You will use it later.

We can read a sentence using a double arrow several ways. We could say "$a \cdot b = 0$ *if and only if* $a = 0$ or $b = 0$".

We could also say

"If $a \cdot b = 0$ then $a = 0$ or $b = 0$, *and conversely*."

Perhaps you noticed the words "if and only if" before. We have used them in definitions. They mean that we are talking about a conditional sentence *and* its converse both being true.

EXERCISES

1. Simplify:

a. ${}^{-}3 \cdot 0$

b. $\frac{4}{3} \cdot {}^{-}5 \cdot 0$

c. ${}^{-}4265 \cdot \frac{12}{99} \cdot 0 \cdot \frac{3}{23}$

d. $\frac{3}{15} \cdot {}^{-}17 \cdot (7 + {}^{-}7)$

e. $({}^{-}\frac{14}{23} + \frac{3}{19}) \cdot ({}^{-}1.5 + \frac{3}{2}) \cdot {}^{-}25$

f. $\left(\frac{49}{{}^{-}\frac{27}{3}} + \frac{75}{12}\right) \cdot (18.3 - 18.3)$

2. Solve these equations for rational numbers.

a. $2 \cdot y = 0$

b. $x \cdot {}^{-}\frac{3}{2} = 0$

c. ${}^{-}\frac{165}{3} \cdot z = 0$

d. ${}^{-}37 \cdot \frac{14}{5} \cdot t = 0$

e. $\frac{2}{3} \cdot x = 1$

f. $p \cdot {}^{-}5 = 5$

Let's Learn a Shortcut

To multiply a number by 101:

By the distributive property of multiplication over addition, for any number n,

$$101 \cdot n = (100 + 1) \cdot n, \quad \text{or} \quad 100 \cdot n + 1 \cdot n.$$

So, to multiply a number by 101 we can multiply by 100 and then add the number. This is easy to do mentally.

To multiply a number by 102:

By the distributive property of multiplication over addition, for any number n,

$$102 \cdot n = (100 + 2) \cdot n, \quad \text{or} \quad 100 \cdot n + 2 \cdot n.$$

So, to multiply a number by 102 we can multiply by 100 and then add twice the number. This is also easy to do mentally.

To multiply a number by 99:

Multiply the number by 100 and then subtract the number. Why?

To multiply a number by 98:

Multiply by 100 and subtract twice the number. Why?

Try these:

1. 101 × 27
2. 102 × 23
3. 99 × 17
4. 98 × 25
5. 101 × 81
6. 102 × 55
7. 101 ×310
8. 102 × 225
9. 99 × 611

10 98. ×350

LONG DIVISION WITH RATIONAL NUMBERS

Suppose you divide a positive number by a negative number. The quotient will be negative. Why? If you divide a negative number by a negative number the quotient will be positive. Why? If we divide a negative number by a positive number, will the quotient be positive or negative?

You know how to perform long division with numbers of arithmetic, so it will be easy to find quotients of rational numbers. You simply divide their absolute values and then decide whether the quotient is positive or negative.

Example: Divide 7420 by $^-35$.

First, divide $|7420|$ by $|^-35|$:

$$\begin{array}{r} 212 \\ 35\overline{)7420} \\ \underline{70} \\ 42 \\ \underline{35} \\ 70 \\ \underline{70} \end{array}$$

The absolute value of the quotient is 212. We are dividing a positive number by a negative one. Therefore the quotient is negative. The answer is $^-212$.

EXERCISES

Divide:

1. 9267 by $^{-}27$
2. $^{-}21{,}168$ by 42
3. $^{-}9734$ by $^{-}31$
4. 45,156 by $^{-}212$
5. 140.99 by $^{-}2.3$
6. $^{-}12.792$ by $^{-}.41$

When we studied the number system of arithmetic we took a good look at long division. Do you remember the basic idea? We make a *guess* at a partial quotient. Then we multiply and subtract. Then we make another guess, multiply and subtract, and so on. Finally we add the partial quotients.

Example:

Divide 7420 by 35.

1. We guess 10. Multiply and subtract.

   Now notice that $35 \cdot 10 + 7070$ is the dividend, 7420.

$$\begin{array}{r} 10 \\ 35\overline{)7420} \\ \underline{350} \\ 7070 \end{array}$$

2. We guess 100, multiply and subtract.

   Notice that $35 \cdot (10 + 100) + 3570$ is the dividend, 7420.

$$\begin{array}{r} 10 + 100 \\ 35\overline{)7420} \\ \underline{350} \\ 7070 \\ \underline{3500} \\ 3570 \end{array}$$

3. We guess 100, multiply and subtract.

   Notice that $35 \cdot (10 + 100 + 100) + 70$ is again the dividend.

$$\begin{array}{r} 10 + 100 + 100 \\ 35\overline{)7240} \\ \underline{350} \\ 7070 \\ \underline{3500} \\ 3570 \\ \underline{3500} \\ 70 \end{array}$$

4. We guess 2 (this is not exactly a guess), multiply and subtract.

$$
\begin{array}{r@{}r}
 & \underline{10 + 100 + 100 + 2} \\
35) & 7420 \\
 & \underline{350} \\
 & 7070 \\
 & \underline{3500} \\
 & 3570 \\
 & \underline{3500} \\
 & 70 \\
 & \underline{70} \\
 & 0
\end{array}
$$

Now, since
$35 \cdot (10 + 100 + 100 + 2) = 7420$,
we have the quotient. It is 212.

At each step, when we multiply 35 by the sum of the partial quotients and then add the remainder, we get 7420, the dividend.

This is not a very useful method for doing long division, but it shows the basic principle of it.

Just for fun, let's see if the same method would work for rational numbers. Now we can guess positive numbers or negative numbers.

Example:

Divide 1107 by ${}^-41$.

1. We guess 10, multiply and subtract.
Notice that ${}^-41 \cdot 10 + 1517 = 1107$.

$$
\begin{array}{r@{}r}
 & \underline{10} \\
{}^-41) & 1107 \\
 & \underline{{}^-410} \\
 & 1517
\end{array}
$$

2. We guess ${}^-20$ this time.

$$
\begin{array}{r@{}r}
 & \underline{10 + {}^-20} \\
{}^-41) & 1107 \\
 & \underline{{}^-410} \\
 & 1517 \\
 & \underline{820} \\
 & 697
\end{array}
$$

Notice that
${}^-41 \cdot (10 + {}^-20) + 697 = 1107$.

3. We guess ${}^-20$ again.

$$
\begin{array}{r@{}r}
 & \underline{10 + {}^-20 + {}^-20} \\
{}^-41) & 1107 \\
 & \underline{{}^-410} \\
 & 1517 \\
 & \underline{820} \\
 & 697 \\
 & \underline{820} \\
 & {}^-123
\end{array}
$$

Notice that
${}^-41 \cdot (10 + {}^-20 + {}^-20) + {}^-123 = 1107$.

4. We guess 3.

$$\begin{array}{r|l} & 10 + {}^{-}20 + {}^{-}20 + 3 \\ \hline {}^{-}41 & 1107 \\ & \underline{{}^{-}410} \\ & 1517 \\ & \underline{820} \\ & 697 \\ & \underline{820} \\ & {}^{-}123 \\ & \underline{{}^{-}123} \\ & 0 \end{array}$$

Since
${}^{-}41 \cdot (10 + {}^{-}20 + {}^{-}20 + 3) = 1107$,
the quotient is ${}^{-}27$. Try this yourself, with the following exercises, just for fun.

EXERCISES

Divide, by making successive guesses:

1. 989 by 23
2. ${}^{-}713$ by 31
3. 966 by ${}^{-}23$
4. ${}^{-}792$ by ${}^{-}24$
5. 15.6 by ${}^{-}1.2$
6. ${}^{-}.384$ by ${}^{-}.12$

DO YOU KNOW THESE TERMS?

To show that you do know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

fulcrum (252)
torque (253)
law of the lever (254)
even integer (258)
odd integer (258)
standard factorization (259)
phrase (261)
equivalent phrases (261)
identity (262)

CHAPTER SUMMARY

Applying the system of rational numbers (251)
The law of the lever (252)
Even and odd integers (258)
Complete factorization of integers (259)
Number phrases for rational numbers (261)
Proving phrases equivalent (262)
Another distributive property (267)
Equivalent fractional expressions (271)
Long division with rational numbers (278)

REVIEW EXERCISES

1. Name the multiplicative identity for integers.

2. Only two integers have multiplicative inverses. Name the integers and the inverse of each.

3. Factor $^{-}12$ as many ways as possible. (Do not use $^{-}1$ or 1 more than once.)

4. List the prime integers between $^{-}1$ and 10.

5. Tell whether each of the following integers is prime or composite:
   a. 8
   b. 10
   c. 7
   d. 43
   e. 51
   f. 91
   g. 81
   h. 0
   i. 1

6. Draw a Venn diagram to show the intersection of the set of even integers between $^{-}11$ and 5 and the set of integers between $^{-}30$ and $^{-}10$.

7. Is the set of odd integers closed under addition? Explain.

8. Is the set of odd integers closed under multiplication? Explain.

9. Write standard factorizations for each of the following integers:
   a. $^{-}6$
   b. 24
   c. $^{-}21$
   d. $^{-}144$
   e. 96

10. When are two number phrases equivalent?

11. What symbol do we use to show that two phrases are equivalent?

12. How do we know that $x + {}^{-}y$ is equivalent to $x - y$?

13. Prove that $^{-}(^{-}x) \equiv x$.

14. Which of the following sentences are true?
    a. $^{-}(x + 1) \equiv {}^{-}x + {}^{-}1$
    b. $^{-}(^{-}y^2) \equiv y^2$
    c. $^{-}2 \cdot x + 3 \cdot x \equiv 2 \cdot x + {}^{-}3 \cdot x$
    d. $x \cdot y \equiv {}^{-}x \cdot {}^{-}y$
    e. $^{-}(2x - y) \equiv y - 2x$

15. Write the converse of $a = 0$ or $b = 0 \rightarrow a \cdot b = 0$.

16. Solve these equations. They refer to the system of rational numbers:
    a. $^{-}\frac{5}{6} \cdot x = 1$
    b. $x \cdot (\frac{3}{4} + {}^{-}\frac{7}{8}) = 0$
    c. $a \cdot {}^{-}\frac{6}{7} = \frac{6}{7}$
    d. $(1 + \frac{3}{5}) + b = 0$
    e. $\dfrac{c}{\frac{5}{6} + {}^{-}\frac{8}{9}} = 0$

17. Graph the following number sentences:
    a. $x + y = 2$
    b. $y = 5x$
    c. $x = {}^-y$
    d. $2y = x$
    e. ${}^-3x + y = 0$

18. Find the square roots of the following rational numbers:
    a. 225
    b. 10,000
    c. 625
    d. 64
    e. 144

19. Mr. Brown borrowed \$500 from a loan company which charges $\frac{1}{2}\%$ of the unpaid balance per month. If he repaid the money in two monthly payments of \$250 each plus interest, how much interest did he pay on the loan?

20. Mary bought a new dress and coat at a clearance sale. All items were reduced $33\frac{1}{3}\%$. Find the total cost of her purchase if the original prices of the dress and coat were \$15 and \$60 respectively, and there was a 3% sales tax.

21. In each of the following insert one of these symbols '>', '<', '=', in order to make a true sentence.
    a. ${}^-4 + \frac{5}{6} \qquad {}^-\frac{5}{6} + 4$
    b. ${}^-\frac{1}{2} \cdot \left(\frac{5}{8} + {}^-\frac{3}{4}\right) \qquad {}^-\frac{1}{2} \cdot {}^-\frac{3}{4} + {}^-\frac{5}{16}$
    c. $\dfrac{\frac{3}{4} + \frac{1}{2} \cdot {}^-\frac{3}{2}}{\frac{5}{16} - \frac{3}{7} + \frac{9}{10}} \qquad \dfrac{\frac{7}{8} + \frac{5}{6}}{\frac{3}{4} + \frac{9}{7}}$
    d. $\dfrac{\sqrt{36} - \sqrt{16}}{5} \qquad \dfrac{5}{\sqrt{36} - \sqrt{16}}$
    e. $(2 \cdot 3) \bmod 5 \qquad 1 \bmod 5$

22. Factor, using the distributive property.
    a. $2 \cdot 4 + 2 \cdot 5$
    b. ${}^-3 \cdot 5 + 7 \cdot {}^-3$
    c. $10 + 15$
    d. $ab + ac$
    e. $3ax + 6a^2x^2$

23. Find the perimeter of a square whose area is 625 sq. ft.

READING LIST

University of Maryland Mathematics Project. *Mathematics for the Junior High School, Second Book.* College Park: Univ. of Md., 1959. pp. 13-46.

CHAPTER SEVEN

Solving Equations

$$a=b \rightarrow a+c=b+c$$

$$a \cdot x=b$$

$$a=b \rightarrow a \cdot c=b \cdot c$$

Let us recall what an equation is. Any number sentence with = for its verb is an equation. An equation like $a = b$ says that 'a' and 'b' are two names of the same number. Some equations are true and some are false. Some equations with variables are neither true nor false. For example,

$$x + \frac{3}{2} = \frac{5}{9}$$

is neither true nor false. A sentence like this is sometimes called an "open sentence." If we replace 'x' by the name of a number, the equation will then become true or false (but not both). We speak of the number named as a "replacement," even though we replace 'x' by a name and not a number. When we find a replacement which makes a sentence true, we say that the number named is a *solution* of that sentence. The set of all replacements making a number sentence true is called the *solution set* of that sentence. When we find the solution set of a sentence, we say we have *solved* it.

Solving equations is important. For many problems in mathematics, science and other fields, we need to solve equations. See how well you can guess and use your imagination to solve the equations in the following exercises. All of the equations in this chapter refer to the system of rational numbers.

EXERCISES

See how many of these equations you can solve.

1. a. $y + 2 = 4$
b. $2 \cdot y = {}^{-}6$
c. ${}^{-}x = \frac{3}{2}$
d. ${}^{-}\frac{4}{3} = {}^{-}t$
e. $x - \frac{1}{4} = \frac{1}{2}$
f. $\frac{1}{2} \cdot x = {}^{-}12$
g. $\frac{3}{4} - p = p$
h. ${}^{-}6 + q = 6$
i. $2 - r = 2$
j. $3 \cdot z - 4 = 8$
k. $\frac{3}{2} - y = \frac{1}{2}$
l. $6 + {}^{-}x = 4$

2. a. ${}^{-}7 = x$
b. ${}^{-}x + 2 = {}^{-}3$
c. ${}^{-}3 \cdot x = \frac{3}{2}$
d. $4 \cdot {}^{-}x = {}^{-}4 \cdot x$
e. $\frac{16}{5} \cdot y = y$
f. $7 \cdot y - {}^{-}2 = 23$
g. $20 = 2 \cdot {}^{-}z + 5$
h. ${}^{-}\frac{1}{2} - \frac{1}{2} \cdot z = {}^{-}2$
i. ${}^{-}\frac{9}{5} + \frac{5}{9} \cdot p = {}^{-}\frac{9}{5}$
j. ${}^{-}(\frac{2}{3} \cdot x) = {}^{-}\frac{2}{3} \cdot x$
k. ${}^{-}\frac{3}{7} \cdot x + \frac{1}{7} = \frac{4}{7}$
l. $\frac{7}{9} - {}^{-}3 \cdot y = \frac{16}{9}$
m. $x + \frac{1}{4} = x + \frac{8}{5}$
n. $x + 1 + x = 2 \cdot x - 1$

3. a. $x = |{}^{-}2|$
b. $|x| = 2$
c. $|y| = {}^{-}2$
d. $y^2 = 4$
e. $x^2 = {}^{-}9$
f. $|x| + 1 = 2$
g. $|p| = p$
h. $|p| = {}^{-}p$
i. $|x|^2 = x^2$
j. $|x|^2 = {}^{-}x^2$
k. $|y|^2 = ({}^{-}y)^2$
l. $|y^2| = y^2$
m. $x^2 = x$
n. $x^2 = {}^{-}x$

4. a. $2 \cdot y + 3 \cdot y = 5$
b. $\frac{1}{3} \cdot y - {}^{-}\frac{1}{3}y = \frac{2}{3}$
c. $\frac{3}{5} \cdot x + {}^{-}\frac{2}{5} = \frac{1}{5} \cdot x$
d. ${}^{-}\frac{16}{5} \cdot p - \frac{2}{5} \cdot p = 0$
e. $\frac{1}{9} \cdot {}^{-}x + \frac{2}{3} = \frac{2}{9} \cdot x$
f. $.8 \cdot z + 1.2 \cdot z = {}^{-}16$
g. ${}^{-}\frac{8}{9} = \frac{1}{3} \cdot z + {}^{-}z$
h. $\frac{3}{5} = \frac{1}{5} \cdot p - {}^{-}p$
i. $\frac{3}{4} \cdot p + {}^{-}\frac{1}{2} \cdot p = {}^{-}\frac{1}{4}$
j. $\frac{8}{7} \cdot x + {}^{-}\frac{2}{7} \cdot x = 6$

5. a. $\frac{9}{5} \cdot y - \frac{3}{7} \cdot y = \frac{3}{8}$
b. $1.3 \cdot y = {}^{-}13.7 \cdot y - 8.6$
c. $\frac{13}{37} \cdot x - {}^{-}\frac{12}{49} \cdot x = \frac{13}{85}$
d. $3 \cdot x^2 + 39 = 37 - {}^{-}2 \cdot x$

Some equations are easy to solve. You can tell at once what the solution set is. Others are more difficult to solve. In the above exercises there may have been some equations which you could not solve.

Remember that solving equations is important. Let's look for some principles we can use to help solve equations.

Review Practice

1. Multiply and simplify:
   - a. $\frac{3}{5} \cdot \frac{2}{3}$
   - b. $\frac{4}{9} \cdot {}^{-}\frac{3}{5}$
   - c. ${}^{-}\frac{12}{5} \cdot {}^{-}\frac{10}{11}$
   - d. $\frac{3}{8} \cdot {}^{-}\frac{2}{3} \cdot \frac{4}{5}$
   - e. ${}^{-}\frac{9}{5} \cdot \frac{10}{9} \cdot {}^{-}\frac{3}{8}$
   - f. ${}^{-}\frac{4}{5} \cdot {}^{-}\frac{11}{6} \cdot {}^{-}\frac{3}{4}$
   - g. ${}^{-}\frac{1}{2} \cdot \frac{2}{3} \cdot {}^{-}\frac{3}{8}$
   - h. $\frac{15}{16} \cdot {}^{-}\frac{4}{5} \cdot \frac{12}{5}$

2. Add, or subtract, and simplify:
   - a. $\frac{2}{5} + {}^{-}\frac{4}{5}$
   - b. ${}^{-}\frac{9}{8} + \frac{3}{4}$
   - c. $\frac{4}{7} + {}^{-}\frac{3}{5}$
   - d. $\frac{5}{12} + \frac{7}{18}$
   - e. $\frac{7}{15} - \frac{11}{25}$
   - f. $\frac{7}{9} - {}^{-}\frac{7}{12}$
   - g. $\frac{15}{36} + {}^{-}\frac{17}{48}$
   - h. $\frac{7}{13} + {}^{-}\frac{4}{26} + \frac{8}{39}$

EQUATIONS LIKE $x + a = b$

We have solved equations such as $x + 1 = 2$. Many of them are easy to solve. Some are not so easy. For the harder ones we would like to find a method.

7.1 Let's Explore

1. Solve these equations. Look for a pattern. Make up some more equations if you need to.
   - a. $x + 2 = 5$
   - b. $y + 4 = 2$
   - c. $m + 4 = 0$
   - d. $q + 3 = {}^{-}1$
   - e. $z + {}^{-}2 = 5$
   - f. $y + {}^{-}3 = 0$
   - g. $x + {}^{-}2 = {}^{-}3$
   - h. $a + \frac{1}{2} = \frac{3}{2}$
   - i. $p + {}^{-}\frac{3}{5} = \frac{1}{5}$
   - j. $q + {}^{-}\frac{4}{9} = {}^{-}\frac{5}{9}$

2. What pattern do you see? Explain it.

3. Try your idea. See if it works to solve these equations:
   - a. $y + {}^{-}\frac{3}{2} = \frac{5}{4}$
   - b. $x + \frac{8}{7} = \frac{4}{3}$
   - c. $z + {}^{-}\frac{12}{11} = {}^{-}\frac{5}{6}$
   - d. $x + \frac{3}{8} = {}^{-}\frac{13}{5}$

You have discovered a "rule" or *principle*, by which you can solve a certain type of equation. We can describe this type of equation as being like:

$$x + a = b$$

Here, we understand that 'x' is a variable, and 'a' and 'b' represent specific numbers. If ‘a’ represents ‘5’ and ‘b’ represents ‘$^-7$’, for example, then $x + a = b$ is the equation $x + 5 = {}^-7$.

Does your “rule” say to subtract a from b? Did your principle work for exercise 3? How could you prove that you are right—that your rule will work all the time?

7.2 Let’s Explore

1. In the equation $x + a = b$, replace ‘x’ by ‘$b - a$’.
2. Use the definition of subtraction to find a phrase equivalent to ‘$b - a$’.
3. Replace ‘$b - a$’ in the left side of your equation of exercise 1 by the phrase equivalent to it.
4. Use the associative property of addition to write another phrase equivalent to that phrase.
5. Simplify the phrase as much as possible. Tell what properties of rational numbers you used.
6. The number represented by the left side of $(b - a) + a = b$ can be represented most simply by ________.
7. Is it the same number represented by the right hand side?
8. Is $b - a$ a solution of $x + a = b$? Why?

You should have proved your principle. To solve an equation like $x + a = b$, subtract a from b. In other words, the equations $x + a = b$ and $b - a = x$ have the same solution set.

To solve any equation like $\mathbf{x} + \mathbf{a} = \mathbf{b}$*, where* ‘$\mathbf{x}$’ *is a variable and* ‘$\mathbf{a}$’ *and* ‘$\mathbf{b}$’ *represent specific numbers, subtract* $\mathbf{a}$ *from* $\mathbf{b}$*.*

EXERCISES

Solve these equations.

1. $x + \frac{^-2}{14} = \frac{5}{21}$
2. $y + 14.63 = {}^-75.02$

3. $z + {}^{-}102.6 = {}^{-}312.17$

4. $x + {}^{-}\frac{17}{36} = {}^{-}\frac{11}{48}$

5. $y + (3.21 \times 10^{-3}) = 5.35 \times 10^{-3}$

6. $p + {}^{-}25\% = 13\%$

Suppose you want to solve the equation:

$${}^{-}\tfrac{3}{2} = x + \tfrac{4}{5}$$

It is not exactly like those we have been studying. How is it different? Can you see that if it were reversed it would be just like those we have been studying? It can be solved the same way. (Remember, any equation is reversible.)

There are some other equations slightly different from $x + a = b$, but which can be solved in the same way.

7.3 Let's Explore

1. Look at the equation ${}^{-}\frac{3}{2} + x = \frac{9}{5}$.
   a. How is it different from an equation like $x + a = b$?
   b. Look at the left side of the equation. Use the commutative property of addition to write an equivalent expression.
   c. Write an equation with that phrase for the left side and '$\frac{9}{5}$' for the right side.
   d. Solve the equation you wrote for part c.
   e. Is the solution set of the equation of part c also the solution set of ${}^{-}\frac{3}{2} + x = \frac{9}{5}$? Why?

2. How can we solve an equation like $a + x = b$?

3. Look at the equation $x - \frac{2}{3} = \frac{4}{5}$.
   a. How is it different from an equation like $x + a = b$?
   b. Look at the left side of the equation. Use the definition of subtraction for rational numbers to write a phrase equivalent to it.
   c. Write an equation with that phrase for the left side and '$\frac{4}{5}$' for the right side.
   d. Solve the equation you wrote for part c.
   e. Is the solution set of the equation of part c also the solution set of $x - \frac{2}{3} = \frac{4}{5}$? Why?

4. How can we solve an equation like $x - a = b$?

You have shown how to solve equations like $a + x = b$ and $x - a = b$. For any such equation we can find another with the same solution set. The other equation is of the type $x + a = b$, and we already know how to solve that type.

EXERCISES

Solve these equations.

1. $\frac{3}{4} + y = 5$
2. $\frac{5}{9} = x + {}^{-}\frac{2}{3}$
3. $z + {}^{-}\frac{4}{5} = \frac{3}{8}$
4. ${}^{-}\frac{3}{5} = {}^{-}\frac{8}{7} + p$
5. $y - {}^{-}\frac{12}{7} = {}^{-}\frac{18}{5}$
6. ${}^{-}4.2 + x = 18.6$
7. $z - 143.6 = {}^{-}17.8$
8. $t - {}^{-}1.6 = \frac{3}{10}$

EQUATIONS LIKE $a \cdot x = b$

You have solved equations like $a \cdot x = b$. Here, again, we understand that ‘x’ is a variable, and ‘a’ and ‘b’ represent specific numbers. An example of this type of equation is $3 \cdot x = {}^{-}5$. Now we wish to find a method for solving such equations.

7.4 Let's Explore

1. Solve these equations. Look for a pattern. Make up some more equations if you need to.
   a. $3 \cdot x = 6$
   b. $4 \cdot x = 16$
   c. $9 \cdot y = 9$
   d. ${}^{-}6 \cdot y = {}^{-}18$
   e. ${}^{-}5 \cdot z = 30$
   f. $2 \cdot z = \frac{1}{2}$
   g. $\frac{1}{3} \cdot x = 4$
   h. ${}^{-}3 \cdot x = \frac{5}{6}$
2. What pattern do you see? Explain it.
3. Try your idea. See if it works for these equations:
   a. ${}^{-}\frac{1}{5} \cdot x = \frac{4}{7}$
   b. $\frac{4}{3} \cdot y = {}^{-}9$
   c. ${}^{-}1.25 \cdot z = {}^{-}3.75$
   d. ${}^{-}\frac{4}{5} \cdot z = {}^{-}\frac{13}{21}$

You have discovered a "rule", or *principle*, by which you can solve equations like $a \cdot x = b$. Does your rule say to divide b by a? Did your rule work for exercise 3 above? How could you prove that your rule will always work?

7.5 Let's Explore

1. In the equation $a \cdot x = b$, replace ‘x’ by $\frac{\text{‘}b\text{’}}{a}$.

2. Simplify the left side of your equation.
3. The number represented by the left side of $a \cdot \left(\frac{b}{a}\right) = b$ can be represented most simply by ________.
4. Is it the same number represented by the right hand side?
5. Is $\frac{b}{a}$ a solution of $a \cdot x = b$? Why?

You should have proved your principle. To solve an equation like $a \cdot x = b$, divide b by a. In other words, the equations $a \cdot x = b$ and $\frac{b}{a} = x$ (where $a \neq 0$) have the same solution set.

Any equation like $\boldsymbol{a \cdot x = b}$*, where* $\boldsymbol{a} \neq 0$*,*
can be solved by dividing **b** *by* **a***.*

EXERCISES

Solve these equations.

1. $3 \cdot x = \frac{14}{3}$
2. $^-3 \cdot y = {^-\frac{8}{5}}$
3. $\frac{4}{5} \cdot z = {^-\frac{7}{3}}$
4. $^-\frac{17}{12} \cdot p = 4 + \frac{1}{2}$
5. $^-1.3 \cdot t = .026$
6. $13\% \cdot x = {^-.39}$

When we studied equations like $x + a = b$, we found that we could use the same principle, even with the equation changed a bit. We learned how to solve equations like $a + x = b$ and $b = x + a$, etc. You have probably already figured out that your principle for solving equations like $a \cdot x = b$ will still work, even if the equation is a bit different. How would you solve equations like these?

1. $x \cdot a = b$
2. $b = a \cdot x$
3. $b = x \cdot a$

You can divide b by a in each case, just like you do for equations like $a \cdot x = b$. Why?

EXERCISES

Solve these equations.

1. $x \cdot \frac{5}{3} = {}^{-}\frac{2}{7}$
2. $1.2 = {}^{-}3.6 \cdot y$
3. ${}^{-}\frac{3}{8} = {}^{-}\frac{5}{9} \cdot z$
4. $14\% = t \cdot 280$
5. $z \cdot \frac{3}{5} = {}^{-}.6$
6. $3.2 \cdot x = {}^{-}64$

EQUATIONS LIKE $a \cdot x + b \cdot x = c$

Let's find a method for solving equations like:

$$a \cdot x + b \cdot x = c$$

Here, again, we understand that 'a', 'b' and 'c' represent specific numbers, and that 'x' is a variable. An example of this type of equation is:

$$4 \cdot x + {}^{-}2 \cdot x = 8$$

7.6 Let's Explore

1. Solve these equations. Look for a pattern. Make up some more equations if you need to.
   - a. $2 \cdot x + 3 \cdot x = 5$
   - b. $\frac{1}{2} \cdot y + \frac{1}{2} \cdot y = 6$
   - c. $\frac{1}{2} \cdot z + \frac{1}{4} \cdot z = 9$
   - d. $4 \cdot x + 2 \cdot x = 12$
   - e. $3 \cdot y + 4 \cdot y = {}^{-}7$
   - f. $\frac{3}{2} \cdot z + \frac{1}{2} \cdot z = {}^{-}8$
   - g. $9 \cdot x + 2 \cdot x = 22$
   - h. ${}^{-}9 \cdot x + {}^{-}2 \cdot x = 22$
2. What pattern do you see? Explain it.
3. If you have not yet seen a pattern, copy and complete this table.

$2 \cdot x + 3 \cdot x = 5$	$2 \cdot x + 3 \cdot x \equiv (2+3) \cdot x \equiv 5 \cdot x$	$5 \cdot x = 5$	$x = \frac{5}{5}$
$9 \cdot x + 2 \cdot x = 22$	$9 \cdot x + 2 \cdot x \equiv (9+2) \cdot x \equiv 11 \cdot x$	$11 \cdot x = 22$	$x = \frac{22}{11}$
$4 \cdot x + 2 \cdot x = 12$			
${}^{-}9 \cdot x + {}^{-}2 \cdot x = 22$			
$3 \cdot y + 4 \cdot y = {}^{-}7$			
$\frac{1}{2} \cdot z + \frac{1}{4} \cdot z = 9$			
$\frac{3}{2} \cdot z + \frac{1}{2} \cdot z = {}^{-}8$			
$\frac{1}{2} \cdot y + \frac{1}{2} \cdot y = 6$			

4. What pattern do you see? Explain it.

5. Try your idea. See if it works for these equations.
   a. ${}^{-}3 \cdot x + \frac{1}{2} \cdot x = 5$
   b. $\frac{2}{3} \cdot y + {}^{-}\frac{5}{3} \cdot y = {}^{-}\frac{3}{5}$
   c. $\frac{11}{12} \cdot z + {}^{-}\frac{3}{4} \cdot z = 3$
   d. ${}^{-}\frac{4}{5} \cdot x + {}^{-}\frac{3}{10} \cdot x = {}^{-}11$
   e. $3.41 \cdot x + {}^{-}2.01 \cdot x = 2.8$
   f. ${}^{-}8.3 \cdot y + 6.1 \cdot y = {}^{-}44$

You have probably discovered a "rule", or principle, by which you can solve equations like $a \cdot x + b \cdot x = c$. You probably used the distributive property. By doing so you can find a phrase equivalent to '$a \cdot x + b \cdot x$'. In other words, you *factor*. Then you have an equation like this:

$$(a + b) \cdot x = c$$

This equation is a type you have already studied. It is like $a \cdot x = b$. To find a solution, you divide c by $a + b$. In other words, the solution is $\frac{c}{a + b}$, if $a + b \neq 0$.

How could we prove that this rule always works? In other words, is $\frac{c}{a + b}$ always a solution of $a \cdot x + b \cdot x = c$, when $a + b \neq 0$?

Let's think about the equation $a \cdot x + b \cdot x = c$, where '$x$' is a variable, and '$a$', '$b$' and '$c$' represent specific rational numbers.

1. Replace 'x' by '$\frac{c}{a + b}$'.

2. Use the distributive property to find an expression equivalent to the left side of your equation.

3. Is the expression you just wrote equivalent to

$$\frac{a + b}{a + b} \cdot c\text{ ?} \quad \text{Why?}$$

4. Simplify the expression further.

5. Is $\frac{c}{a + b}$ a solution of $a \cdot x + b \cdot x = c$? Why?

Now you should have proved your principle. To solve an equation like $a \cdot x + b \cdot x = c$, where $a + b \neq 0$, first factor, to get $(a + b) \cdot x = c$. Then divide c by $a + b$.

To solve any equation like $\mathbf{a} \cdot \mathbf{x} + \mathbf{b} \cdot \mathbf{x} = \mathbf{c}$, *where* $\mathbf{a} + \mathbf{b} \neq 0$, *first factor, to obtain*

$$(\mathbf{a} + \mathbf{b}) \cdot \mathbf{x} = \mathbf{c}.$$

Then divide $\mathbf{c}$ *by the factor* $\mathbf{a} + \mathbf{b}$.

EXERCISES

Solve these equations.

1. $5 \cdot x + 2 \cdot x = 21$
2. $4 \cdot y + {}^{-}2 \cdot y = 6$
3. $9 \cdot x + 3 \cdot x = 48$
4. $8 \cdot y + {}^{-}11 \cdot y = 3$
5. ${}^{-}4 \cdot x + {}^{-}2 \cdot x = 12$
6. $10 \cdot y + {}^{-}12 \cdot y = 16$
7. $7 \cdot x + {}^{-}2 \cdot x = 12$
8. ${}^{-}6 \cdot y + 2 \cdot y = 6$
9. $\frac{2}{3} \cdot x + \frac{4}{3} \cdot x = \frac{8}{3}$
10. $5 \cdot y + {}^{-}\frac{1}{2} \cdot y = {}^{-}9$
11. $\frac{7}{3} \cdot z + {}^{-}\frac{2}{3} \cdot z = 5$
12. $\frac{8}{5} \cdot t + {}^{-}\frac{3}{10} \cdot t = \frac{26}{5}$
13. $1.32 \cdot x + {}^{-}1.02 \cdot x = {}^{-}6$
14. ${}^{-}8 \cdot x + {}^{-}3 \cdot x = \frac{22}{7}$
15. $18.2 \cdot y + {}^{-}6.2 \cdot y = {}^{-}6$
16. ${}^{-}\frac{17}{3} \cdot y + {}^{-}\frac{5}{6} \cdot y = \frac{26}{5}$

As with the other equations we have studied, our principle will work, even if we change the type of equation slightly. Suppose we try to solve:

$$2 \cdot x + x \cdot 5 = {}^{-}14$$

The left side is not quite like those we have been studying. But the following phrase is equivalent to the left side. Why?

$$2 \cdot x + 5 \cdot x$$

When we write an equation with this expression for the left side, it will be just like those we have been studying.

Suppose we try to solve:

$$x \cdot {}^{-}3 + x \cdot 4 = 3$$

We can use the expression $x \cdot ({}^{-}3 + 4)$ instead of the left side, because it is equivalent to the left side. Then we would divide 3 by $({}^{-}3 + 4)$.

The equation

$$17 = 5 \cdot y + 3 \cdot y$$

is like those we have been studying except that it is reversed. We can solve it using the same principle.

Now let's look at this equation.

$$7 \cdot x - 3 \cdot x = {}^{-}24$$

On the left side we have a subtraction sign instead of an addition sign. We could use the expression

$$7 \cdot x + {}^{-}(3 \cdot x)$$

which is equivalent to the left side. Or we could remember that we have a distributive property of multiplication over subtraction, and factor '$7 \cdot x - 3 \cdot x$' as '$(7 - 3) \cdot x$'. Then we would divide ${}^{-}24$ by $(7 - 3)$.

In all of these cases we still use the same basic principle. We factor; then we divide.

EXERCISES

Solve these equations.

1. $6 \cdot x - 2 \cdot x = 8$
2. $9 \cdot y - 11 \cdot y = 6$
3. $4 = 8 \cdot x + {}^{-}10 \cdot x$
4. $y \cdot 4 - y \cdot 2 = 7$
5. ${}^{-}x \cdot 2 + 5 \cdot x = {}^{-}9$
6. $9 \cdot y + {}^{-}(2 \cdot y) = 7$
7. $x \cdot 3 + {}^{-}\frac{1}{2} \cdot x = {}^{-}10$
8. $7 \cdot y - 5 \cdot y = \frac{3}{5}$
9. ${}^{-}7 = \frac{3}{5} \cdot x - x \cdot \frac{3}{10}$
10. $\frac{28}{5} = 8 \cdot y - {}^{-}6 \cdot y$
11. ${}^{-}3 = {}^{-}4 \cdot t + {}^{-}2 \cdot t$
12. $x \cdot 5 - x \cdot {}^{-}2 = {}^{-}13$
13. ${}^{-}y \cdot 2 + \frac{6}{5} \cdot y = \frac{5}{4}$
14. ${}^{-}6 = \frac{3}{5} \cdot x + {}^{-}\frac{2}{5} \cdot x$
15. $3 \cdot x + 2 \cdot x + 8 \cdot x = 169$
16. $9 \cdot x + 2 \cdot x - 5 \cdot x = 54$
17. $8 \cdot x + x = 27$
18. $7 \cdot x - 1 \cdot x = 42$

GENERAL PRINCIPLES FOR SOLVING EQUATIONS

We have studied several types of equations and how to solve them. There are many more types. If we had to remember a "rule", or principle, for solving each different kind, this would be too complicated. Perhaps we can find some principles that will work for many different kinds of equations.

THE ADDITION PRINCIPLE

7.7 Let's Explore

1. a. Is this sentence true? $3 + 1 = 4$.

b. If 2 is added to $3 + 1$, what is the result? If 2 is added to 4, what is the result?
Is $(3 + 1) + 2 = 4 + 2$ true?

c. If 6 is added to $3 + 1$, the result can be named ‘$(3 + 1) + 6$’. If 6 is added to 4, the result can be named ‘$4 + 6$’.
Is $(3 + 1) + 6 = 4 + 6$ true?

d. Now let's add 0. What number is named by ‘$(3 + 1) + 0$’? by ‘$4 + 0$’?
Is $(3 + 1) + 0 = 4 + 0$ true?

e. Now let's add $^{-}6$. What number is named by ‘$(3 + 1) + {}^{-}6$’? by ‘$4 + {}^{-}6$’?
Is $(3 + 1) + {}^{-}6 = 4 + {}^{-}6$ true?

f. Now let's add $^{-}1$. What number is named by ‘$(3 + 1) + {}^{-}1$’? by ‘$4 + {}^{-}1$’?
Is $(3 + 1) + {}^{-}1 = 4 + {}^{-}1$ true?

2. a. Is this sentence true? $9 = 10 + {}^{-}1$.
b. Suppose we add 2. What number is named by ‘$9 + 2$’? by ‘$(10 + {}^{-}1) + 2$’?
Is $9 + 2 = (10 + {}^{-}1) + 2$ true?
c. Let's add $^{-}10$. What number is named by ‘$9 + {}^{-}10$’? by ‘$(10 + {}^{-}1) + {}^{-}10$’?
Is $9 + {}^{-}10 = (10 + {}^{-}1) + {}^{-}10$ true?
d. Let's add 1. What number is named by ‘$9 + 1$’? by ‘$(10 + {}^{-}1) + 1$’?
Is $9 + 1 = (10 + {}^{-}1) + 1$ true?
e. Let's add $^{-}9$. What number is named by ‘$9 + {}^{-}9$’? by ‘$(10 + {}^{-}1) + {}^{-}9$’?
Is $9 + {}^{-}9 = (10 + {}^{-}1) + {}^{-}9$ true?
f. Let's add $^{-}12$. What number is named by ‘$9 + {}^{-}12$’? by ‘$(10 + {}^{-}1) + {}^{-}12$’?
Is $9 + {}^{-}12 = (10 + {}^{-}1) + {}^{-}12$ true?

3. What principle or pattern do you see? Explain it.

You should have discovered that whenever we have an equation, like $a = b$, we can find another one by adding. We add some number to a. We also add the same number to b. Then we have a new equation $a + c = b + c$. If the first equation, $a = b$, is true, then a and b are the same number. When we add c to this number we get *just one* answer, because addition is an operation and the set of rational numbers is closed under addition. The result of the addition can be called

'$a + c$' or '$b + c$.' These symbols must represent the same number. Then the new equation must be true if the first one is true. This principle holds for any equation in the system of rational numbers.

> *Whenever we get a new equation by adding, it will be true if the original equation is true.*
>
> *Or:*
>
> *If any equation in rational numbers,* $\boldsymbol{a} = \boldsymbol{b}$*, is true, then* $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{c}$ *will be true, for any rational number* $\boldsymbol{c}$*.*

Another way of saying this is "$a = b \rightarrow a + c = b + c$." This principle is simple, but it is important. It will help us to solve many equations. To see how it works, let's try it on some simple equations. We already know a way to solve equations like $x + a = b$, but let's see how our new principle works for such equations.

7.8 Let's Explore

1. Let's think about the equation $x + 2 = 11$.
   a. Get a new equation by adding $^{-}2$.
   b. Simplify both sides of your new equation.
   c. What is the solution set of your new equation?
2. You have proved that *if* $x + 2 = 11$, *then* $x = 9$.
   This means that any replacement for 'x' making $x + 2 = 11$ true also makes $x = 9$ true. Does this mean that a replacement making $x = 9$ true also must make $x + 2 = 11$ true?
3. Check to see whether the solution of $x = 9$ is also a solution of $x + 2 = 11$.
4. Let's think about the equation $x + {}^{-}\frac{3}{5} = \frac{4}{3}$.
   a. Find a new equation by adding $\frac{3}{5}$.
   b. Simplify both sides of your new equation.
   c. What is the solution set of your new equation?
   d. Check to see if the solution of your new equation is also a solution of the original equation.
5. Let's think about the equation $x - 4.8 = 9.1$.
   a. Find a new equation by adding 4.8.
   b. Simplify both sides of your new equation.
   c. Find the solution set of your new equation.
   d. Check to see if the solution of your new equation is also a solution of the original one.

6. Let's think about the equation $\frac{9}{2} - x = 3$.
    - a. Find a new equation by adding x.
    - b. Simplify the left side.
    - c. Find a new equation by adding ${}^{-}3$.
    - d. Simplify both sides of your new equation.
    - e. Find the solution set of this equation.
    - f. Check to see if the solution of this equation is also a solution of the original equation.

You have seen how to use the *addition principle*. To solve an equation, we find some new ones by adding. Usually we add the additive inverse of something named in the equation so as to get the additive identity, 0. The idea is to work down to a very simple equation, whose solution set can be seen easily. Then we check to see whether we have a solution of the original equation.

Here are some examples showing how to write your work to keep it neat.

Example 1:

Solve:

$$\begin{aligned} x + {}^{-}\tfrac{3}{2} &= \tfrac{5}{6} \\ x + {}^{-}\tfrac{3}{2} + \tfrac{3}{2} &= \tfrac{5}{6} + \tfrac{3}{2} \\ x + 0 &= \tfrac{5}{6} + \tfrac{3}{2} \\ x &= \tfrac{5}{6} + \tfrac{9}{6} \\ x &= \tfrac{14}{6} \\ x &= \tfrac{7}{3} \end{aligned}$$

There is one and only one solution of this last equation. It is $\frac{7}{3}$.
Check:

$$\begin{array}{c|c} x + {}^{-}\frac{3}{2} \;\; = & \frac{5}{6} \\ \hline \frac{7}{3} + {}^{-}\frac{3}{2} & \\ \frac{7}{3}\cdot\frac{2}{2} + {}^{-}\frac{3}{2}\cdot\frac{3}{3} & \\ \frac{14}{6} + {}^{-}\frac{9}{6} & \\ \frac{5}{6} & \frac{5}{6} \end{array}$$

The solution is $\frac{7}{3}$

Example 2:

Solve:

$$\begin{aligned} {}^{-}5 - x &= 2 \\ {}^{-}5 - x + x &= 2 + x \end{aligned}$$

$$\begin{aligned} {}^-5 - 0 &= 2 + x \\ {}^-5 &= 2 + x \end{aligned}$$

The solution set of this last equation is $\{{}^-7\}$.

Check:

$$\begin{array}{c|c} \multicolumn{2}{c}{{}^-5 - x = 2} \\ \hline {}^-5 - {}^-7 & 2 \\ {}^-5 + 7 & \\ 2 & 2 \end{array}$$

The solution set of the original equation is $\{{}^-7\}$.

EXERCISES

Solve these equations using the addition principle.

1. $x + \frac{4}{3} = \frac{5}{3}$
2. $y + {}^-\frac{1}{3} = \frac{5}{6}$
3. $z - \frac{3}{5} = \frac{4}{5}$
4. $18.1 + x = {}^-3.2$
5. ${}^-\frac{3}{4} + y = \frac{1}{2}$
6. ${}^-1.62 + y = {}^-3.84$
7. $6 - x = 4$
8. ${}^-\frac{4}{5} - x = \frac{3}{10}$
9. $8 - x - \frac{1}{2} = 13 + \frac{1}{4} + \frac{1}{2} + \frac{3}{4}$
10. ${}^-x + 21.2 = {}^-6.13 - 1.02$
11. $\frac{11}{2} = 12 + z$
12. ${}^-14.2 = x + 18.3$
13. ${}^-14.6 = 7 - y$
14. $\frac{3}{4} = {}^-x + \frac{3}{8}$
15. $4.56 + y = {}^-8.2$
16. ${}^-17.2 = 4.5 + y$
17. $y - \frac{12}{5} = {}^-\frac{6}{10}$
18. $\frac{16}{7} - y = {}^-\frac{3}{14}$
19. $\frac{4}{9} = {}^-x + \frac{2}{3}$
20. ${}^-17.5 = 8.1 - x$

THE MULTIPLICATION PRINCIPLE

You discovered the *addition principle* and learned to use it in solving equations. Perhaps you have suspected that there is another principle like it for multiplication.

7.9 Let's Explore

1. a. Is this sentence true? $3 \cdot 5 = 15$.

b. If $3 \cdot 5$ is multiplied by 2, what is the result? If 15 is multiplied by 2, what is the result? Is $(3 \cdot 5) \cdot 2 = 15 \cdot 2$ true?

c. If $3 \cdot 5$ is multiplied by 4, the result can be named '$4 \cdot (3 \cdot 5)$'. If 15 is multiplied by 4, the result can be named '$4 \cdot 15$'. Is $4 \cdot (3 \cdot 5) = 4 \cdot 15$ true?

d. Now let's multiply by $\frac{1}{3}$. What number is named by '$\frac{1}{3} \cdot (3 \cdot 5)$'? by '$\frac{1}{3} \cdot 15$'? Is $\frac{1}{3} \cdot (3 \cdot 5) = \frac{1}{3} \cdot 15$ true?

e. Let's multiply by $\frac{1}{5}$. What number is named by '$(3 \cdot 5) \cdot \frac{1}{5}$'? by '$15 \cdot \frac{1}{5}$'? Is $(3 \cdot 5) \cdot \frac{1}{5} = 15 \cdot \frac{1}{5}$ true?

f. Let's multiply by $^{-}2$. What number is named by '$^{-}2 \cdot (3 \cdot 5)$'? by '$^{-}2 \cdot 15$'? Is $^{-}2 \cdot (3 \cdot 5) = {^{-}2} \cdot 15$ true?

g. Let's multiply by 0. What number is named by '$0 \cdot (3 \cdot 5)$'? by '$0 \cdot 15$'? Is $0 \cdot (3 \cdot 5) = 0 \cdot 15$ true?

2. What principle or pattern do you see? Explain it.

You should have discovered that whenever we have an equation like $a = b$, we can find another one by multiplying. We multiply a by some number. We also multiply b by the same number. Then we have a new equation $a \cdot c = b \cdot c$. If the first equation $a = b$ is true, then a and b are the same number. When we multiply this number by c we get *just one* answer because multiplication is an operation and the set of rational numbers is closed under multiplication. The result of the multiplication can be called '$a \cdot c$' or '$b \cdot c$'. These symbols must represent the same number. Then the new equation must be true if the first one is true. This principle holds for any equation referring to the system of rational numbers.

Whenever we get a new equation by multiplying, it will be true if the original equation is true.

Or:

If any equation in rational numbers, $\boldsymbol{a} = \boldsymbol{b}$, *is true, then* $\boldsymbol{a} \cdot \boldsymbol{c} = \boldsymbol{b} \cdot \boldsymbol{c}$ *will be true, for any rational number* $\boldsymbol{c}$.

Another way of saying this is "$a = b \rightarrow a \cdot c = b \cdot c$, for any rational number c."

This principle is much like the addition principle. It is also important in solving equations. To see how it works, let's try it on some simple equations.

7.10 Let's Explore

1. Let's think about the equation $4 \cdot x = 31$.
   a. Get a new equation by multiplying by $\frac{1}{4}$.
   b. Simplify both sides of your new equation.
   c. What is the solution set of your new equation?

2. You have proved that *if* $4 \cdot x = 31$ is true, *then* $x = \frac{31}{4}$ is true. This means that any replacement for 'x' making $4 \cdot x = 31$ true also makes $x = \frac{31}{4}$ true.

   Does this mean that a replacement making $x = \frac{31}{4}$ true also must make $4 \cdot x = 31$ true?

3. Check to see whether the solution of $x = \frac{31}{4}$ is also a solution of $4 \cdot x = 31$.

4. Let's think about the equation $^{-}\frac{4}{3} \cdot y = \frac{8}{9}$.
   a. Get a new equation by multiplying. Multiply by $^{-}\frac{3}{4}$.
   b. Simplify both sides of your new equation.
   c. What is the solution set of your new equation?
   d. Check to see if the solution of your new equation is also a solution of the original equation.

You have seen how to use the *multiplication principle*. To solve an equation, we find some new ones by multiplying. Usually we multiply by the reciprocal of something named in the equation so as to get the multiplicative identity, 1. The idea is to work down to a very simple equation, whose solution set can be seen easily. Then we check to see whether we have a solution of the original equation.

Example:

Solve:

$$\begin{aligned} 3 \cdot x &= 25 \\ \tfrac{1}{3} \cdot 3 \cdot x &= \tfrac{1}{3} \cdot 25 \\ 1 \cdot x &= \tfrac{25}{3} \end{aligned}$$

There is one and only one solution of this last equation. It is $\frac{25}{3}$.

Check:

$$\begin{array}{r|l} 3 \cdot x & = 25 \\ \hline 3 \cdot \frac{25}{3} & \\ \frac{3}{3} \cdot 25 & \\ 25 & 25 \end{array}$$

The solution is $\frac{25}{3}$.

EXERCISES

Solve these equations using the multiplication principle.

1. $4 \cdot x = 8$
2. ${}^{-}5 \cdot y = 10$
3. $x \cdot 7 = 3$
4. ${}^{-}6 \cdot y = {}^{-}11$
5. $5 \cdot x = {}^{-}7$
6. $y \cdot {}^{-}3 = {}^{-}5$
7. $3 \cdot x = {}^{-}18$
8. $y \cdot {}^{-}5 = 21$
9. $z \cdot \frac{2}{3} = {}^{-}\frac{5}{3}$
10. $\frac{y}{{}^{-}5} = {}^{-}27$
11. $\frac{{}^{-}y}{4} = \frac{1}{2}$
12. $z \cdot {}^{-}\frac{3}{5} = \frac{6}{5}$
13. $\frac{x}{{}^{-}\frac{1}{2}} = {}^{-}\frac{3}{2}$
14. ${}^{-}1.2 = y \cdot \frac{3}{5}$
15. ${}^{-}\frac{4}{3} = {}^{-}z \cdot \frac{8}{5}$
16. $\frac{14}{11} = {}^{-}(2 \cdot x)$
17. ${}^{-}y \cdot \frac{5}{2} = {}^{-}\frac{3}{4}$
18. ${}^{-}(z \cdot {}^{-}\frac{3}{4}) = \frac{5}{9}$
19. • $\frac{3}{x} = \frac{5}{8}$
20. • $\frac{5}{2} = \frac{6}{{}^{-}x}$

• Hint for 19 and 20: Multiply by x or ${}^{-}x$.

Some of the above exercises can be made simpler by recalling a few simple facts about equivalent phrases. Let's recall that ${}^{-}1 \cdot x \equiv {}^{-}x$. This means that the two expressions, '${}^{-}1 \cdot x$' and '${}^{-}x$' will name the same number for all sensible replacements (and *all* replacements are sensible in this case). Therefore we can replace '${}^{-}x$' by '${}^{-}1 \cdot x$' in any equation without changing the solution set. Here is an example showing how this may simplify solving an equation.

Example:

Solve:

$$\begin{aligned} {}^{-}y \cdot \tfrac{2}{3} &= {}^{-}\tfrac{5}{9} \\ {}^{-}1 \cdot y \cdot \tfrac{2}{3} &= {}^{-}\tfrac{5}{9} && \text{(replacing '${}^{-}y$' by '${}^{-}1 \cdot y$')} \\ {}^{-}\tfrac{2}{3} \cdot y &= {}^{-}\tfrac{5}{9} && \text{(simplifying)} \\ {}^{-}\tfrac{3}{2} \cdot {}^{-}\tfrac{2}{3} \cdot y &= {}^{-}\tfrac{3}{2} \cdot {}^{-}\tfrac{5}{9} && \text{(multiplying by ${}^{-}\tfrac{3}{2}$)} \\ y &= \tfrac{5}{6} && \text{(simplifying)} \end{aligned}$$

Check to see if $\frac{5}{6}$ is a solution of the original equation.

Sometimes, in solving an equation you may work down to an equation with '${}^{-}x$' alone on one side. For example, you might have an equation like this

$${}^{-}x = \tfrac{3}{2}$$

You know that its solution is ${}^{-}\frac{3}{2}$. You have already handled cases like this by adding x, but we may also use the multiplication principle, as follows: Replace '${}^{-}x$' by '${}^{-}1 \cdot x$' and then multiply by ${}^{-}1$.

Example:

Solve:

$$\begin{aligned} {}^{-}x &= \tfrac{3}{2} \\ {}^{-}1 \cdot x &= \tfrac{3}{2} && \text{(replacing '${}^{-}x$' by '${}^{-}1 \cdot x$')} \\ {}^{-}1 \cdot {}^{-}1 \cdot x &= {}^{-}1 \cdot \tfrac{3}{2} && \text{(multiplying by ${}^{-}1$)} \\ x &= {}^{-}\tfrac{3}{2} && \text{(simplifying)} \end{aligned}$$

It is easy to see that ${}^{-}\frac{3}{2}$ is a solution of the original equation.

Whenever you find an equation like ${}^{-}x = b$, you will know that its solution is ${}^{-}b$, the additive inverse of b. If you should forget this, you could always use the addition principle (and add x) or the multiplication principle (and multiply by ${}^{-}1$).

Let's recall that ${}^{-}(a \cdot b) \equiv {}^{-}a \cdot b$. Since this is true, we can replace a symbol like '${}^{-}(a \cdot b)$' in any equation by the corresponding symbol '${}^{-}a \cdot b$' without changing the solution set. For example, if we have '${}^{-}(4 \cdot y)$' in an equation we can replace it by '${}^{-}4 \cdot y$' without changing the solution set.

Example:

Solve:

$$\begin{aligned} \tfrac{5}{9} &= {}^{-}(\tfrac{2}{3} \cdot x) \\ \tfrac{5}{9} &= {}^{-}\tfrac{2}{3} \cdot x && \text{(replacing '$(^{-}\tfrac{2}{3} \cdot x)$' by '${}^{-}\tfrac{2}{3} \cdot x$')} \\ {}^{-}\tfrac{3}{2} \cdot \tfrac{5}{9} &= {}^{-}\tfrac{3}{2} \cdot {}^{-}\tfrac{2}{3} \cdot x && \text{(multiplying by ${}^{-}\tfrac{3}{2}$)} \\ {}^{-}\tfrac{5}{6} &= x && \text{(simplifying)} \end{aligned}$$

Check to see if ${}^{-}\frac{5}{6}$ is a solution of the original equation.

Let's recall that ${}^{-}x \cdot y \equiv x \cdot {}^{-}y$. This fact is also useful in solving equations. We could replace a symbol like '${}^{-}x \cdot y$' by the corresponding symbol '$x \cdot {}^{-}y$' whenever it will help. For example, we could replace '${}^{-}x \cdot \frac{4}{3}$' by '$x \cdot {}^{-}\frac{4}{3}$'.

Example:

Solve:

$$\begin{aligned} \frac{4}{5} &= {}^{-}y \cdot \frac{10}{7} \\ \frac{4}{5} &= y \cdot {}^{-}\frac{10}{7} && \text{(replacing '}{}^{-}y \cdot \tfrac{10}{7}\text{' by '}y \cdot {}^{-}\tfrac{10}{7}\text{')} \\ \frac{4}{5} \cdot {}^{-}\frac{7}{10} &= y \cdot {}^{-}\frac{10}{7} \cdot {}^{-}\frac{7}{10} && \text{(multiplying by } {}^{-}\tfrac{7}{10}\text{)} \\ {}^{-}\frac{14}{25} &= y && \text{(simplifying)} \end{aligned}$$

Check to see if ${}^{-}\frac{14}{25}$ is a solution of the original equation.

For any equation you may use any of these ideas when they help. There are usually several different ways to solve an equation.

EXERCISES

Solve these equations.

1. ${}^{-}\frac{4}{5} = {}^{-}x \cdot \frac{2}{3}$
2. ${}^{-}(\frac{4}{5} \cdot y) = {}^{-}\frac{3}{5}$
3. ${}^{-}x = {}^{-}3.21$
4. $\frac{4}{3} \cdot {}^{-}y = {}^{-}\frac{7}{2}$
5. ${}^{-}(x \cdot {}^{-}\frac{4}{9}) = \frac{8}{3}$
6. ${}^{-}y \cdot \frac{4}{3} = {}^{-}\frac{9}{5}$
7. $\frac{9}{5} = {}^{-}y$
8. ${}^{-}\frac{3}{8} = {}^{-}x \cdot {}^{-}\frac{4}{5}$
9. ${}^{-}y \cdot \frac{3}{5} = {}^{-}\frac{4}{5}$
10. $\frac{6}{11} = {}^{-}(\frac{5}{11} \cdot y)$
11. $\frac{4}{5} = {}^{-}(z \cdot {}^{-}\frac{2}{3})$
12. $\frac{5}{9} \cdot {}^{-}z = {}^{-}\frac{2}{3}$
13. ${}^{-}\frac{4}{5} = {}^{-}y$
14. ${}^{-}(\frac{3}{7} \cdot {}^{-}x) = {}^{-}\frac{3}{14}$

Let's Learn a Shortcut

Have you been practicing the shortcuts you know? Here is another one which may be new to you.

To multiply by 5:

We know that $5 = \frac{10}{2}$, so to multiply by 5 we can multiply by $\frac{10}{2}$. This means, also, that we can multiply by 10 (this is easy to do mentally) and then divide by 2 (this is easy, too).

Example:

$$5 \times 312 = \frac{10}{2} \times 312$$

Multiply by 10: 3120
Take half: 1560

Try these:

1. 5×21
2. 5×44
3. 5×82
4. 5×802
5. 5×422
6. 5×2048
7. 5×8262
8. 5×4868

Make up your own shortcuts for multiplying by 50 or 500, and then practice them.

USING THE PRINCIPLES TOGETHER

Now that you know how to use the addition and multiplication principles, let's see how to use them together.

7.11 Let's Explore

1. Let's think about the equation $3 \cdot x + \frac{1}{2} = \frac{5}{2}$.
   a. Get a new equation by adding $^-\frac{1}{2}$. Simplify both sides.
   b. Now get a new equation by multiplying by $\frac{1}{3}$. Simplify it.
   c. What is the solution set of your last equation?
   d. Check to see whether the solution is also a solution of the original equation.

2. Let's think about the equation $\frac{3}{8} = \frac{1}{2} + {^-\frac{4}{3}} \cdot y$.
   a. Get a new equation by adding $^-\frac{1}{2}$. Simplify it.
   b. Get a new equation by multiplying by $^-\frac{3}{4}$. Simplify it.
   c. What is the solution set of your last equation?
   d. Check to see whether the solution is also a solution of the original equation.

You have seen how to use the addition and multiplication principles together. You have used them to solve equations like $a \cdot x + b = c$. Both multiplication and addition are mentioned in the equation. We usually get a new equation by adding, and then another one by multiplying. Then, of course, we must check to see if we have a solution of the original equation.

EXERCISES

Solve these equations.

1. $6 \cdot x + 5 = 17$
2. $4 \cdot y + 7 = {}^-5$
3. $6 + 2 \cdot x = 10$
4. $^-5 + 3 \cdot y = 7$
5. $9 = 7 \cdot x - 5$
6. $^-6 + 4 \cdot y = 10$
7. $5 \cdot x + \frac{2}{3} = {}^-\frac{5}{6}$
8. $^-\frac{2}{3} \cdot x + \frac{1}{2} = {}^-\frac{7}{8}$
9. $\frac{3}{8} + 2 \cdot y = {}^-7$
10. $^-\frac{4}{5} + 5 \cdot y = \frac{2}{3}$
11. $z \cdot \frac{4}{3} - 3 = {}^-\frac{8}{9}$
12. $z \cdot {}^-\frac{5}{3} + {}^-\frac{2}{3} = \frac{5}{6}$
13. $^-\frac{7}{6} = \frac{4}{3} \cdot x + 2$
14. $2 = {}^-\frac{2}{3} + x \cdot \frac{3}{5}$
15. $4 \cdot {}^-y + \frac{3}{8} = \frac{9}{3}$
16. $\frac{3}{4} = 6 \cdot {}^-z - \frac{3}{8}$
17. $2 \cdot x + 5 \cdot x + 3 = 38$
18. $^-22 = 5 \cdot x + {}^-7 \cdot x + 2$

You have already discovered a "rule" or principle for solving equations like $a \cdot x + b \cdot x = c$. Recall that rule says to factor, to get $(a + b) \cdot x = c$. We can then multiply by the reciprocal of $a + b$, using the multiplication principle. Now let's see how we can use our new principles to solve equations like $a \cdot x = b \cdot x + c$. In equations like this, at least one phrase containing a variable is on each side of the equation.

7.12 Let's Explore

1. Let's think about the equation $5 \cdot x = 3 \cdot x + \frac{3}{2}$.
   a. Find a new equation using the addition principle. Add $^-(3 \cdot x)$.
   b. Find an expression equivalent to '$^-(3 \cdot x)$' which has no parentheses.
   c. In your new equation, replace '$^-(3 \cdot x)$' by the equivalent expression.
   d. Look at the left side of the equation now. Can you factor? If so, do it. If you cannot, go back to part b and find a different expression so that you can factor.
   e. Simplify.
   f. Use the multiplication principle to get a new equation. Find its solution set.
   g. Check to see if you have a solution of the original equation.

2. Let's think about the equation $4 \cdot x + \frac{1}{2} = 6 \cdot x + {}^{-}\frac{1}{4}$.
   a. Find a new equation using the addition principle. Add $\frac{1}{4}$.
   b. Find another equation using the addition principle. Add ${}^{-}(4 \cdot x)$.
   c. Find an expression equivalent to '${}^{-}(4 \cdot x)$' which has no parentheses.
   d. In your new equation, replace '${}^{-}(4 \cdot x)$' by the equivalent expression.
   e. Factor and simplify.
   f. Use the multiplication principle to get a new equation. Find its solution set.
   g. Check to see if you have a solution of the original equation.

You have now seen how to solve equations like $a \cdot x = b \cdot x + c$, where there is at least one phrase containing the variable on each side of the equation. We use the addition principle to get an equation which has the variables all on one side. Then we factor. After that, we use the multiplication principle.

There are some equations, slightly different from $a \cdot x = b \cdot x + c$ for which the same method works.

Example:

Solve:

$$\begin{aligned} 2 \cdot y + {}^{-}2 &= 3 \cdot y + \tfrac{1}{2} \\ {}^{-}(2 \cdot y) + 2 \cdot y + {}^{-}2 &= {}^{-}(2 \cdot y) + 3 \cdot y + \tfrac{1}{2} \\ {}^{-}2 &= {}^{-}2 \cdot y + 3 \cdot y + \tfrac{1}{2} \\ {}^{-}2 + {}^{-}\tfrac{1}{2} &= {}^{-}2 \cdot y + 3 \cdot y + \tfrac{1}{2} + {}^{-}\tfrac{1}{2} \\ {}^{-}\tfrac{5}{2} &= {}^{-}2 \cdot y + 3 \cdot y \\ {}^{-}\tfrac{5}{2} &= 1 \cdot y \end{aligned}$$

Check:

$$\begin{array}{c|c} 2 \cdot y + {}^{-}2 & 3 \cdot y + \tfrac{1}{2} \\ \hline 2 \cdot {}^{-}\tfrac{5}{2} + {}^{-}2 & 3 \cdot {}^{-}\tfrac{5}{2} + \tfrac{1}{2} \\ {}^{-}5 + {}^{-}2 & {}^{-}\tfrac{15}{2} + \tfrac{1}{2} \\ {}^{-}7 & {}^{-}\tfrac{14}{2} \\ & {}^{-}7 \end{array}$$

The solution set is $\{{}^{-}7\}$.

By now you should see that our two principles can be used for many different types of equations. Using them is better than learning a rule for each different kind of equation. We cannot solve *every* kind of equation using these two principles, but we can solve a great many kinds.

EXERCISES

Solve these equations.

1. $4 \cdot x = 3 \cdot x + 7$
2. $5 \cdot y - 2 = 3 \cdot y$
3. ${}^-8 \cdot x = 4 \cdot x + {}^-6$
4. $5-4 \cdot y+2 \cdot y=3 \cdot y+{}^-5$
5. $3 \cdot x-5 \cdot x+7 \cdot x=4+3 \cdot x$
6. $3 \cdot y = 2 - 5 \cdot y$
7. $9 \cdot x = 2 \cdot x + 12$
8. $5 \cdot x + \frac{3}{4} = 9 \cdot x$
9. ${}^-2 \cdot y = \frac{4}{3} + 3 \cdot y$
10. ${}^-\frac{1}{2} \cdot y + \frac{5}{6} = \frac{3}{4} \cdot y$
11. $2 \cdot z - 5 = \frac{3}{2} \cdot z + 2$
12. $\frac{2}{3} - 3 \cdot z + 2 \cdot z = {}^-3 \cdot z + \frac{1}{3}$
13. $x - \frac{1}{2} = \frac{3}{4} + 4 \cdot x$
14. ${}^-x + \frac{4}{3} \cdot x - 2 = \frac{4}{3} - \frac{7}{3} \cdot x$
15. $1.2 \cdot y - 2 \cdot y + 3 = {}^-3.2y + 1.6$
16. $4.1 \cdot y - 5.5 + {}^-2.1 \cdot y = 13.1 - 6.2 \cdot y + {}^-1.1 \cdot y$

Now you have learned general principles for solving equations. You will solve these kinds of equations many times.

When you have an equation to solve, how should you start? Sometimes you can tell the solution just by looking at the equation. If you can, there is no need to use the principles, or write out steps. All you need to do is check to make sure you are right. If you cannot see what the solution is, then use the addition and multiplication principles. But *always* check to make sure you are right, no matter how you find a solution.

EXERCISES

Solve these equations. Use the addition and multiplication principles only when you cannot see at once what the solution is. Check to make sure you are right in all cases.

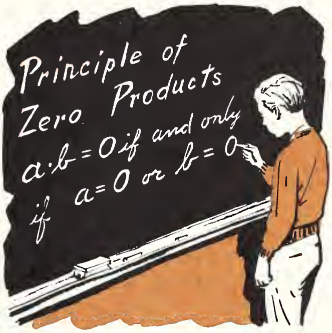
1. $3 \cdot x = 6$
2. $5 = x - 2$
3. $2 + 2 \cdot x = 8$
4. $2 \cdot x + 2 \cdot x = 4$
5. $3 \cdot y - 2 \cdot y = {}^-\frac{1}{2}$
6. ${}^-y = y$
7. $6 - z = 3$
8. ${}^-2 \cdot z + 4 = 2 \cdot z$
9. $x - \frac{3}{2} = \frac{2}{3}$
10. $\frac{4}{5} - x = \frac{3}{8}$
11. ${}^-\frac{3}{8} \cdot y = \frac{1}{4}$
12. $\frac{4}{5} \cdot {}^-y + 3 = \frac{2}{3}$

13. $\frac{3}{5} + {}^{-}x = \frac{5}{9}$
14. ${}^{-}x + \frac{2}{3} = 1$
15. $71 = {}^{-}z - 43$
16. ${}^{-}2 \cdot z + \frac{4}{3} = 2 \cdot z + \frac{4}{3}$
17. $3 \cdot x + {}^{-}7 \cdot x = 8$
18. ${}^{-}3 = \frac{1}{5} \cdot x$
19. $\frac{2}{3} \cdot x + \frac{3}{8} = \frac{1}{3} \cdot x$
20. ${}^{-}(4.2 \cdot x) - 2 \cdot x = {}^{-}x + 10.4$
21. $2 \cdot {}^{-}y = \frac{1}{4} + y$
22. ${}^{-}x - \frac{4}{5} = \frac{3}{10}$
23. ${}^{-}\frac{3}{2} + y = {}^{-}\frac{2}{3}$
24. $\frac{9}{7} - x = \frac{3}{14}$
25. ${}^{-}(\frac{2}{5} \cdot y) + \frac{4}{5} \cdot {}^{-}y = {}^{-}\frac{3}{10}$
26. $35 \cdot {}^{-}y = 70$
27. $\frac{4}{3} \cdot x - x = x - \frac{2}{3}$
28. $\frac{3}{8} \cdot x - {}^{-}(\frac{1}{4} \cdot x) = {}^{-}\frac{5}{8}$
29. $x + \frac{1}{2} = {}^{-}\frac{2}{3} + x$
30. $4 \cdot x + 2 - x = 2 + 3 \cdot x$

THE PRINCIPLE OF ZERO PRODUCTS

In the last chapter you learned an important property of rational numbers. We can now use it to solve some kinds of equations.

Remember that for any rational numbers a and b, $a \cdot b = 0$ if and only if $a = 0$ or $b = 0$. This means that if any factor of a product is zero, then the product is zero. Also that if any product is zero, then at least one of the factors must be zero. How can we use this property to solve equations?


Principle of Zero Products
a·b = 0 if and only if a = 0 or b = 0

7.13 Let's Explore

1. Let's think about the equation $(x - 2) \cdot (x + 2) = 0$.
   a. If it is to be true, what product must be 0?
   b. What are the factors of that product?
   c. Find a replacement which makes $x - 2$ zero.
   d. In the original equation, replace 'x' by '2'. Does 2 make the equation true?
   e. Find a replacement which makes $x + 2$ zero.
   f. In the original equation, replace 'x' by '${}^{-}2$'. Does ${}^{-}2$ make the equation true?
   g. What is the solution set of the original equation?

2. Let's think about the equation $(x + 3) \cdot (x + 2) = 0$.
   a. If it is to be true, what product must be 0?
   b. What are the factors of that product?
   c. Find a replacement which makes $x + 3$ zero.
   d. Is that number a solution of the original equation?
   e. Find a replacement which makes $x + 2$ zero.
   f. Is that number a solution of the original equation?
   g. What is the solution set of the original equation?
3. Can you state a principle for solving equations like $(x - a) \cdot (x + b) = 0$?

You have seen a way to use our important property of rational numbers to solve certain kinds of equations. The equations we have in mind must *always* have '0' on one side. On the other side there must be a factorization. Every replacement which makes a factor zero is a solution of the equation. This is the *principle of zero products.*

Any equation with '0' on one side and a factorization on the other can be solved as follows: Find all the replacements which make a factor zero.

Here is an example of how to write your work, so that it will be neat. That way you will make fewer mistakes.

Solve: $(2 \cdot x - 1) \cdot (x + 4) = 0$

$$\begin{aligned} 2 \cdot x - 1 = 0 \text{ or } x + 4 &= 0 \\ \text{If } 2 \cdot x - 1 &= 0 \\ 2 \cdot x &= 1 \\ x &= \tfrac{1}{2} \\ \text{If } x + 4 &= 0, \\ x &= {}^{-}4 \end{aligned}$$

Check: for $\frac{1}{2}$

$$\begin{array}{c|c} (2 \cdot x - 1) \cdot (x + 4) \quad = & 0 \\ \hline (2 \cdot \frac{1}{2} - 1) \cdot (\frac{1}{2} + 4) & \\ (1 - 1) \cdot (\frac{1}{2} + 4) & \\ 0 \cdot (\frac{1}{2} + 4) & \\ 0 & 0 \end{array}$$

for ${}^-4$	$(2 \cdot x - 1) \cdot (x + 4)$	$=$	0
	$(2 \cdot {}^-4 - 1) \cdot 0$		
	0		0

The solution set is $\{{}^-4, \frac{1}{2}\}$.

EXERCISES

Use the principle of zero products to solve these equations.

1. $(x - 5) \cdot (x + 7) = 0$
2. $(x - \frac{1}{2}) \cdot (x - \frac{2}{3}) = 0$
3. $({}^-5 + y) \cdot (y + \frac{3}{4}) = 0$
4. $(y - {}^-\frac{4}{5}) \cdot ({}^-\frac{3}{8} + y) = 0$
5. $z \cdot (z - 2) = 0$
6. ${}^-z \cdot (3 + z) = 0$
7. $(2 \cdot x + 1) \cdot (\frac{1}{2} \cdot x - 4) = 0$
8. $(- 3 \cdot x + 4) \cdot (\frac{4}{5} \cdot x + 2) = 0$
9. $(y + 2) \cdot (y - 3) \cdot (y - 5) = 0$
10. $(2 \cdot y - 1) \cdot (y + \frac{1}{2}) \cdot (3y - 2) = 0$
11. $(x - 2) \cdot (x - 3) + 1 = 1$
12. $(x - 4) \cdot (x + 2) = 5$
13. $y \cdot y = 0$
14. $(3 \cdot y + 2) \cdot {}^-y = 0$

There are other equations which can be solved by the principle of zero products. They look different from the ones we have been solving.

7.14 Let's Explore

1. Let's think about the equation $(x - 2) \cdot (x + 2) = 0$.
   a. Let's use the distributive property. Find an expression equivalent to the left side. Copy and fill in the blanks:
   $(x - 2) \cdot (x + 2) \equiv (x + \underline{\qquad}) \cdot (x + 2) \equiv$
   $\underline{\qquad} \cdot (x + 2) + \underline{\qquad} \cdot (x + 2) \equiv$
   $\underline{\qquad} \cdot x + \underline{\qquad} \cdot 2 + \underline{\qquad} \cdot x + \underline{\qquad} \cdot 2 \equiv x^2 - \underline{\qquad}$
   b. Now replace the left side of the original equation by the expression equivalent to it.
   c. What is the solution set of the equation you just wrote?

2. Let's think about the equation $(x + 3) \cdot (x + 2) = 0$.
   a. Use the distributive property. Find an expression equivalent to the left side. Copy and fill in the blanks:
   $(x + 3) \cdot (x + 2) \equiv$
   $\underline{\qquad} \cdot (x + 2) + \underline{\qquad} \cdot (x + 2) \equiv$
   $\underline{\qquad} \cdot x + \underline{\qquad} \cdot 2 + \underline{\qquad} \cdot x + \underline{\qquad} \cdot 2 \equiv$
   $x^2 + \underline{\qquad} \cdot x + 6$
   b. Now replace the left side of the original equation by the expression equivalent to it.

You have seen that the principle of zero products might be used to solve equations like:

$$x^2 - 4 = 0 \text{ and}$$
$$x^2 + 5 \cdot x + 6 = 0$$

To use it on $x^2 - 4 = 0$, we would first need to find a factorization of $x^2 - 4$. To use it on $x^2 + 5 \cdot x + 6 = 0$, we would need to find a factorization of $x^2 + 5 \cdot x + 6$.

Perhaps you can see that it is important to learn more about phrases like these. We shall need to learn more about factoring. Then we shall be able to solve more kinds of equations. In the next chapter you will learn more about these kinds of expressions, or phrases.

Review Practice

1. Write standard factorizations of these integers.
    - a. 12
    - b. $^{-}6$
    - c. 45
    - d. $^{-}27$
    - e. $^{-}65$
    - f. 32
    - g. $^{-}81$
    - h. 250
    - i. $^{-}300$
2. Simplify:
    - a. $3^2 \cdot 3^3$
    - b. $4^5 \cdot 4^3$
    - c. $(^{-}2)^5 \cdot (^{-}2)^4$
    - d. $6^7 \cdot 6^{-2}$
    - e. $3^4 \cdot 3^{-4}$
    - f. $5^2 \cdot 5^{-4}$
    - g. $(^{-}6)^3 \cdot (^{-}6)^4$
    - h. $(^{-}7)^4 \cdot (^{-}7)^{-6}$
    - i. $8^4 \cdot 8^5 \cdot 8^{-2}$

DO YOU KNOW THESE TERMS?

To show that you do know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

solution (285)
solution set (285)
principle (287)
addition principle (298)
multiplication principle (301)
principle of zero products (310)

CHAPTER SUMMARY

Equations like $x + a = b$ (287)
Equations like $a \cdot x = b$ (290)
Equations like $a \cdot x + b \cdot x = c$ (292)
Addition principle (295)
Multiplication principle (299)
Using the principles together (305)
The principle of zero products (309)

REVIEW EXERCISES

1. Find solution sets in the system of rational numbers for each of the following equations:

   a. $x + {}^{-}\frac{3}{4} = \frac{5}{2}$
   b. $y + \frac{7}{8} = \frac{1}{4}$
   c. $4 \cdot a = 20$
   d. $\frac{a}{3} = {}^{-}15$
   e. $2 \cdot b + 4 \cdot b = {}^{-}36$
   f. $9 \cdot c - 4 \cdot c = 125$
   g. $\frac{2}{3} \cdot x + 6 = 20$
   h. $\frac{5}{6} = 3x - \frac{2}{3}$
   i. $7y + \frac{3}{4} - 2y = \frac{1}{2}$
   j. $9 - c = 15$
   k. $a + \frac{3}{4} = \frac{3}{4} + a$
   l. $x + \frac{1}{2} = {}^{-}\frac{1}{2} + x$
   m. $(a + 3) \cdot (a - 2) = 0$
   n. $(x - \frac{1}{2}) \cdot (x - \frac{1}{2}) = 0$
   o. $(x - 2) \cdot (x + 3) + 1 = 1$
   p. $(3 \cdot y + 3) \cdot (2 \cdot y + 8) \cdot (4 \cdot y - 2) = 0$
   q. $a \cdot (a + 2) + 3 \cdot (a + 2) = 0$
   r. $(5 \cdot x + 2) \cdot {}^{-}x = 0$
   s. $4 \cdot (3 \cdot x + 1) = 0$
   t. $a \cdot a = 0$

2. List the members of the intersection of the set of rational numbers between 5 and ⁻5 and the set of integers.

3. Draw a concave quadrilateral and show that it is concave.

4. Draw a convex pentagon and show that it is convex.

5. a. Draw a circle.
   b. Draw a diameter in your circle.
   c. Construct the perpendicular bisector of the diameter, extending this bisector to the circle so that it also is a diameter of the circle.
   d. Bisect all the angles with vertex at the center of the circle.
   e. Using the points where the diameters intersect the circle as end points draw line segments, such that no more than 2 segments have the same end point.
   f. What is the name of the polygon you have drawn?

6. Write scientific notation for:

$$34{,}000{,}000 \times .000000000016$$

7. Find the total bill for the following items:

3 cans pears	@ 2 cans for 53¢
2 cans tomatoes	@ 3 cans for 62¢
3 lbs. ground beef	@ 2 lbs. for $1.00
1 can soup	@ 3 cans for 31¢

8. Simplify: (Write exponential notation for your answer.)

a. $\frac{3^3 \cdot 2^2 \cdot 10^5}{10^3 \cdot 3 \cdot 2^7}$

b. $\frac{a^2 \cdot b^3 \cdot c^5}{a^5 \cdot b^3 \cdot c^2}$

c. $\frac{25^3 \cdot a^7 \cdot b^3 \cdot c^{-10}}{5^6 \cdot a^{-2} \cdot b^5 \cdot c^{10}}$

d. $\frac{2a^2 \cdot 3b^5 \cdot 2^3 \cdot a^{-3}}{3^{-3} \cdot a^4 \cdot b^7}$

e. $\frac{3^{-7} \cdot 5^{-10} \cdot x^5 \cdot y^{-10}}{7^5 \cdot 3^{-4} \cdot x^{-2} \cdot y^7}$

9. Which of the following are equivalent phrases?

a. $3x^2 + 6x^2$ and $3 \cdot (x^2 + 2x^2)$

b. $3x^2 + 6x^2$ and $3x^2 \cdot 2x^2$

c. $x \cdot \left(\frac{1}{x} + \frac{2}{x}\right)$ and $1 + 2$

d. $2a + 4a + {}^{-}3a$ and $a \cdot (2 + 4 + {}^{-}3)$

e. $5b + {}^{-}6b$ and $3a + 5b + {}^{-}6b + {}^{-}3a$

10. The ratio of the width of a rectangle to its length is 3:5. If the width is 12 ft., what is the length?

11. Find solution sets in the system of integers for each of the following:

a. $|x| < 5$

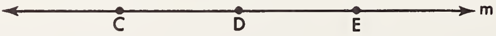
b. $|x| < {}^{-}5$

c. $x = |{}^{-}5|$

d. $x > |{}^{-}5|$

e. $x < |5|$

12.



If C, D and E are points on a line, as shown, describe the following sets:

a. $\overset{\circ\!\!\longrightarrow}{CD} \cap \overset{\circ\!\!\longrightarrow}{DC}$

b. $\overline{CE} \cup m$

c. $\overline{CD} \cup \overline{DE}$

d. $\overrightarrow{DE} \cup \overrightarrow{DC}$

e. $\overset{\circ\!\!\longrightarrow}{DE} \cap \overset{\circ\!\!\longrightarrow}{DC}$

13. Draw examples of each of the following:

a. A non-simple hexagon

b. A regular quadrilateral

c. 2 concentric circles such that the ratio of their radii is 1:2.

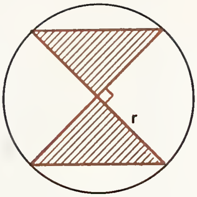
d. A concave triangle

e. A right scalene triangle.

14. If the measures of two of the angles of a triangle are 45° and 55°, what is the measure of the third angle?

15. Prove that the acute angles of a right triangle are complementary angles.

16. Write a formula for finding the area of the shaded region below:



17. Find the area of a circle with a circumference of 6.28 ft. (Use 3.14 for π)

18. Graph each of the following number sentences:
    - a. $a + b = 4$
    - b. $a = b$
    - c. ${}^{-}2a + 2b = 0$
    - d. $3a = b$
    - e. $a = {}^{-}2b$

19. Find the square roots of the following numbers:
    - a. $\frac{1}{16}$
    - b. $\frac{9}{25}$
    - c. 36
    - d. $\frac{625}{10{,}000}$
    - e. 1.44

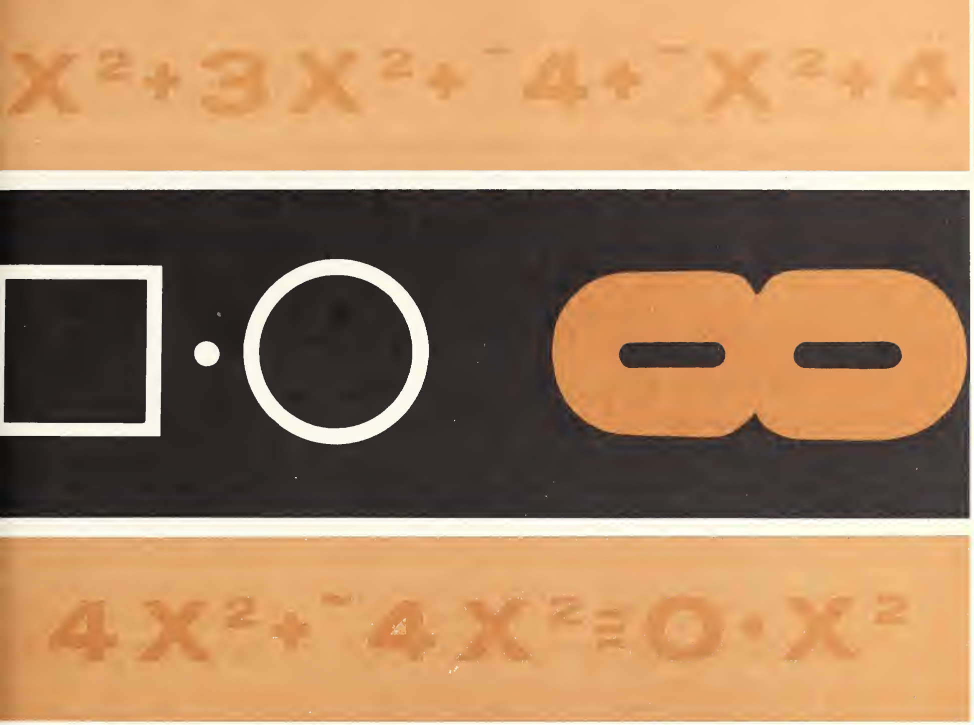
20. What do we mean when we say that the order of the rational numbers is dense?

READING LIST

University of Maryland Mathematics Project. *Mathematics for the Junior High School, Second Book.* College Park: Univ. of Md., 1959. pp. 69-98.

CHAPTER EIGHT

Polynomials



When we studied equations we used symbols like these.

$$2 \cdot x + 3, \qquad {}^{-}5 \cdot x + 4, \qquad 2 \cdot x^2 + 5 \cdot x + 4$$

We found that such expressions are important. We shall now learn more about them.

Numbers can be multiplied or factored, but we have agreed that we do not multiply or factor symbols. In the last chapter, however, it looked as if we needed to define multiplication and factoring for certain phrases. We can do it by making up a new mathematical system. The members of the set will be symbols, and we shall define multiplication and addition for those symbols.

THE MULTIPLICATION DOT

Before we make up our new system let's agree on a way to simplify our writing. We write a dot to mean multiply. The symbol '$3 \cdot 4$' means to multiply 3 by 4. The symbol '$5 \cdot x$' means to multiply 5 by x. In '$5 \cdot x$' we can leave out the dot. We can write '$5x$' and agree that it means the same as '$5 \cdot x$'. We could not leave out the dot in '$3 \cdot 4$'. If we did we would write '34', and this does not mean to multiply 3 by 4. What does it mean?

Let's agree that we can leave out the multiplication dot whenever it doesn't confuse us.

POLYNOMIALS

In our new mathematical system we will include only certain kinds of expressions.

8.1 Let's Explore

1. Numerals for rational numbers are important symbols. We shall want them in our new system. Here are some such symbols: $^{-}3$, 0, $\frac{2}{3}$, $^{-}\frac{3}{5}$. Write ten more symbols like these.

2. Variables are important symbols, and we shall want them in our new system. Write five different variable symbols.

3. Symbols like 'x^3' are also important. They have a variable and an exponent symbol. Write ten more symbols like these.

4. Symbols like '$3y$', '$^{-}2y^4$' and '$\frac{3}{2}x^2$' are also important. They have a numeral and a variable. Some of them have exponent symbols, too. Write ten more symbols like these.

5. Study these symbols.

   $$4x^2 + 3x, \qquad 3y^4 + 5, \qquad ^{-}4x^3 - 3x^2, \qquad 9y^4 + 5y^2 - 3$$

   They contain symbols like you wrote for questions 1-4, joined by plus or minus signs. Write ten more symbols like these.

All of the symbols which you have been writing can be called *polynomials*. When we make up our mathematical system we want it to be useful, but simple. So before we define polynomials, let's explore some more. Perhaps we can simplify things by making some agreements.

Polynomials in one variable. The first agreement we make is to study symbols with just one variable. Later we can study symbols with more than one variable. We shall usually use the letter 'x' for the variable.

Plus and Minus Signs. Some polynomials contain plus signs or minus signs. Can we make an agreement about them?

8.2 Let's Explore

1. Look at the phrase $3x - 2$. Find a phrase equivalent to it having a plus sign: $3x + \underline{\qquad}$.
2. Find a phrase equivalent to '${}^{-}5x^2 - x$' having a plus sign instead of a minus sign: ${}^{-}5x^2 + \underline{\qquad}$.
3. For the phrase $x^3 - 2x$, find an equivalent phrase: $x^3 + \underline{\qquad}x$.
4. Find an expression equivalent to $3x^4 - 2x^5 - 2x$, having plus signs instead of minus signs:
   $3x^4 + \underline{\qquad} + \underline{\qquad}$.
5. What does it mean for phrases to be equivalent?

You have now seen that for any phrase showing subtractions, we can find an equivalent one showing additions. You know this from the definition of subtraction for rational numbers.

This gives us one way of simplifying symbolism. We can consider just those expressions whose parts are joined with *plus* signs. Whenever we have an expression with minus signs, we can replace it with an equivalent one having only plus signs.

EXERCISES

For each expression, find an equivalent one having only plus signs between its parts.

1. $5x - \frac{7}{2}$
2. $4x - {}^{-}3$
3. ${}^{-}5x - x^3$
4. $\frac{2}{3}x^4 + {}^{-}4x$
5. $7x^2 - {}^{-}3x^5$
6. ${}^{-}6x^3 - 2x^7$
7. $4x^3 - {}^{-}3x^5 + 2$
8. ${}^{-}2x^3 + 3x^6 - 4x$
9. $\frac{2}{3}x^4 - {}^{-}2x^3 - 7x^5 - 4x - 2$
10. ${}^{-}\frac{4}{5}x^2 + 2x^5 - 3x^4 + 4 - 5x^3 - {}^{-}6x^2$

"COMBINING" SIMILAR TERMS

In the expressions we have been studying there are parts with plus signs between them. These parts are called *terms*. In the expression

$$4x^3 + {}^{-}2x + \tfrac{5}{2} + {}^{-}7x^3 + 5x$$

the *terms* are: $4x^3$, ${}^{-}2x$, $\frac{5}{2}$, ${}^{-}7x^3$, and $5x$.

In the above expression two of the terms have 'x^3' in them. We say that terms like these are *similar*. There are also two terms with 'x' (with no exponent symbol) in them. These two terms are *similar terms*. If a term has 'x^4' in it, and another has 'x^2' in it, then those terms are not similar. Let's see how we can simplify our problem further by thinking about similar terms.

8.3 Let's Explore

1. Study the expression $3x^2 + 2x + 2x^2$.
    - a. How many terms has it?
    - b. Write them.
    - c. Are any two of them *similar*?
    - d. Is the following expression equivalent to the expression above? Why or why not?
    $$3x^2 + 2x^2 + 2x$$
    - e. Look at the first two terms in part d. Use the distributive property to write an equivalent expression.
    - f. Replace the first two terms of part d by the equivalent expression.
    - g. Is your answer to part f equivalent to the original expression? Why?
2. Study the phrase $4x^3 + 5x + 4x^2 + {}^{-}2x^3 + 5x^2$.
    - a. What are its terms?
    - b. Are some of the terms similar?
    - c. Write an equivalent expression which has similar terms together.
    - d. Use the distributive property to simplify.

You have seen how to use the distributive property to simplify expressions with similar terms. When we do this we say we are "combining" similar terms. We don't actually put symbols together of course. The word *combine* here means to use the distributive property to find an equivalent expression. You will need some practice now, in "combining similar terms."

EXERCISES

Simplify each expression, if possible, by "combining similar terms."

1. $3x^2 + 5x^2$
2. $4x^3 + {}^{-}1x^3 + 2$
3. $5x^4 + 4x^3 + 2x^4$
4. ${}^{-}2x^5 + 4x^3 + x^2$
5. $4x^3 + {}^{-}5x^2 + 3x^2 + 9x^3$
6. $\frac{1}{2}x^4 + 3x^2 + \frac{3}{4}x^4 + {}^{-}5$
7. $3x^4 + 2x^3 + 3 + {}^{-}5$
8. $\frac{4}{3} + 5x^2 + {}^{-}\frac{5}{3}$

9. ${}^{-}5x^3 + 7x^2 + {}^{-}2x^2 + {}^{-}5x^3$
10. $3x^4 + {}^{-}2x^4 + 7x^4 + {}^{-}4x^4$
11. $4x^2 + {}^{-}4x^2$
12. $3x + {}^{-}3x$
13. $4x^5 + 7x^2 + {}^{-}3x^3 + {}^{-}7x^2$
14. $3x^2 + \frac{3}{4} + 7x^2 + {}^{-}\frac{3}{4}$
15. $3x^2 + x^2$
16. $14x^5 + {}^{-}x^5$
17. $3x^5 + 2 + {}^{-}x^5 + 4x^5$
18. $x^2 + 3x^2 + {}^{-}4 + {}^{-}x^2 + 4$
19. $\frac{1}{2}x^5 + {}^{-}x^5 + 3x^2 + {}^{-}x^2$
20. ${}^{-}3x^2 + {}^{-}5x^3 + 8x^4 + 5x^3 + 4$

In the above exercises you have probably found out several important things. Let's list them, to make sure you understand.

1. Terms like '3' and '5', without variables, are *similar*. They may be combined.
2. When we combine similar terms, we may get zero. For example:

   $$4x^2 + {}^{-}4x^2 \equiv 0 \cdot x^2, \text{ or } 0$$

   When this happens, we could write '0' or we could write '$0 \cdot x^2$'. If there are other terms left, we need not write anything. For example:

   $$2x^3 + 4x^2 + {}^{-}4x^2 \equiv 2x^3$$

3. Sometimes we have to supply a '1' or a '${}^{-}1$' to use the distributive property. As examples:

   $$\begin{aligned} 3x^4 + x^4 &\equiv 3x^4 + 1x^4 \\ &\equiv (3 + 1)x^4 \\ &\equiv 4x^4 \end{aligned}$$

   $$\begin{aligned} 4x^5 + {}^{-}x^5 &\equiv 4x^5 + {}^{-}1x^5 \\ &\equiv (4 + {}^{-}1)x^5 \\ &\equiv 3x^5 \end{aligned}$$

Often we use expressions which do not have all plus signs between the parts. We can always find equivalent expressions which do have plus signs. Or, since we have a distributive property of multiplication over *subtraction*, we can combine similar terms without changing to plus signs.

Examples:

$$\begin{aligned} 5x^2 - 2x^2 &\equiv (5 - 2)x^2 \\ &\equiv 3x^2 \end{aligned}$$

$$\begin{aligned} 4x^2 + 3x^4 - 2x^2 - x^4 &\equiv (4 - 2)x^2 + (3 - 1)x^4 \\ &\equiv 2x^2 + 2x^4 \end{aligned}$$

EXERCISES

Simplify each expression, if possible, by combining similar terms.

1. $5x - 2x$
2. $5x^3 - 2x^3 + 2$
3. ${}^{-}4x^4 + 3x^2 - 2x^4$
4. $3x^2 - 5x^2$
5. ${}^{-}3x^3 - 5x^2 - 2x^2 - 2x^3$
6. $6x^4 - 3x^2 - 5x^4$
7. ${}^{-}6x^3 + 2x^5 - 2x^3 - 5x^5$
8. $7x^2 - 5x^3 - 2x^2 - 5x^3$
9. $4x - x$
10. $7x^2 + 3x - x^2 - x$
11. ${}^{-}3x^4 - x^2 + 2x^4 + 6x^2$
12. $3x^2 + 4x^6 - 2x^6 - 3x^2 - 2x^6$
13. $7x^2 + 1 - x^2 + x^3 - 1 - 3x^2$
14. $\frac{3}{2}x^5 + 4x - \frac{1}{2}x^5 - x$
15. $\frac{3}{4}x^3 + 2x^2 - x^3 + \frac{1}{2}x^2$
16. $\frac{2}{3}x^4 - 3 - \frac{1}{2}x + \frac{2}{3}x^4 + \frac{1}{4}x + \frac{1}{4}$

ASCENDING OR DESCENDING ORDER

Now we know how to simplify expressions by combining similar terms. We are working up to a *standard* way of writing expressions. Now let's think about the order of the terms. We know we can change the order of the terms in an expression. By the associative and commutative properties, we get an equivalent expression when we change the order of terms.

Examples:

$$2x^5 + 4x^7 + x^2 + 2x \equiv 4x^7 + 2x^5 + x^2 + 2x$$

$$3 + 4x^5 + 4x^2 + 5x + 3x^3 \equiv 4x^5 + 3x^3 + 4x^2 + 5x + 3$$

Notice how the terms on the right have been arranged. The term with the largest exponent comes first. The term with the next largest exponent comes next, and so on. When expressions are arranged like this we say they are in *descending order*.

Descending order is a standard way of writing expressions. We shall usually arrange them this way. Sometimes the opposite order is also used. That is, the term with the smallest exponent comes first. That kind of order is called *ascending order*.

Almost always we shall use descending order. It is not wrong to use ascending order, or some other order, but descending order will be our *standard* way of writing polynomials.

EXERCISES

1. Arrange each of the following expressions in descending order:
   a. $x + 3x^5 + 4x^3 + 5x^2 + 6x^7 + {}^-2x^4$
   b. $4x^2 + {}^-3 + 7x^5 + 2x^3 + {}^-5x^4$
   c. ${}^-14 + 6x^2 + {}^-9x^5 + 14x^8$
   d. ${}^-17x^9 + 13x^{12} + {}^-47x^3 + \frac{3}{2} + {}^-31x^7 + 25x^4$
2. Combine similar terms and then arrange in descending order:
   a. $3x^2 + 4x^5 + {}^-2x + 3 + {}^-5x^2 + x^5 + {}^-1 + {}^-x$
   b. ${}^-x + \frac{1}{2} + 14x^4 + {}^-2x^3 + {}^-7x + {}^-1 + x^3 + {}^-2x + {}^-4x^4$
   c. $3x^3 + {}^-7x^8 + 2x + {}^-4 + {}^-2x^3 + 7x^8 + 5$
   d. $3x^3 + 5x^5 + 4x^4 + {}^-2x^3 + {}^-5x^5 + 3 + {}^-3x^4 + {}^-2 + {}^-x^4$

Now we are ready to give a definition of *polynomials*. In every polynomial we have one or more terms. In each term we have a numeral. (Sometimes we have to supply a '1' or a '${}^-1$' for this to be so.) Every term has a variable, except perhaps the last one.

A ***polynomial*** *is any expression of the following type:*

$$\square\, x^n + \cdots + \triangle\, x^2 + \diamondsuit\, x + \nabla$$

We understand that in each frame belongs a numeral for a rational number and that ***n*** *is a natural number.*

We sometimes refer to the *degree* of a term. The degree of a term is the exponent of that term. For example the term $5x^4$ is of degree 4. If a term has a variable with no exponent it is of degree 1. If a term has no variable it is of degree zero.

Sometimes we write expressions having minus signs between some terms. We may also call such expressions "polynomials," but the above definition gives the *standard* way of writing them.

The numbers named in the frames above are called the *coefficients* of the terms.

Example:

In the polynomial $3x^4 + {}^-2x^3 + 5x^2 + x + {}^-3$

the *coefficient* of the first term is 3
the *coefficient* of the second term is ${}^-2$

the *coefficient* of the third term is 5
the *coefficient* of the fourth term is 1
the *coefficient* of the fifth term is $^{-}3$

Sometimes coefficients are zero. When this happens we usually do not write the term. We also sometimes say that a term with a zero coefficient is a *missing term.*

Example:

In the polynomial, $4x^5 + {}^{-}2x^3 + 5x + 4$, there is no term with 'x^4'

We can say "the x^4 term (or the *fourth degree* term) is missing,"

or

we can say "the x^4 term (or fourth degree term) has a *zero coefficient.*"

There is also no term with 'x^2'.

We can say "the x^2 term (or *second degree* term) is missing,"

or

we can say "the x^2 term (or second degree term) has a *zero coefficient.*"

If we wish, we can write terms with zero coefficients. Then there would not be any "missing" terms. For example, for the polynomial $3x^2 + 2$, we could write $3x^2 + 0 \cdot x + 2$. Usually we do not write terms with zero coefficients, but it would not be wrong to do so.

Some polynomials have just one term. They are called *monomials.* A polynomial with just two terms is called a *binomial.* A polynomial with just three terms is called a *trinomial.*

EXERCISES

Study these polynomials. Then answer the following questions.

a. $2x^3 + 4x^2 + {}^{-}2$
b. ${}^{-}2x^5$
c. $x^2 + 1$
d. $x^4 + {}^{-}x^2 + 3x + {}^{-}\frac{1}{2}$
e. $9x^4 + 2x^3 + {}^{-}\frac{1}{3}x^2 + {}^{-}x + \frac{9}{5}$
f. $3x^3 + {}^{-}5x + \frac{7}{3}$
g. ${}^{-}4x^2$
h. ${}^{-}x$
i. ${}^{-}\frac{7}{2}x^5 + x$
j. ${}^{-}x^6 + 3x^4 + {}^{-}x^2$
k. $3x^2 + 4x + {}^{-}7$

1. Which of the above polynomials are binomials?
2. Which of the above polynomials are monomials?

3. Which of the above polynomials are trinomials?

4. For a, d, e, f, i and j, name the coefficient of the third degree term.

5. For a, c, d, e, f and g, tell which terms are "missing".

6. Rewrite a, c, d and f, putting in the "missing" terms, with zero coefficients.

7. What is the coefficient of:
   a. the second degree term in d?
   b. the first degree term in h?
   c. the fourth degree term in d?
   d. the first degree term in i?

8. Write expanded numerals for each of the following:
   a. 3204_{10}
   b. 623317_7
   c. 44032_5
   d. 2104325_b

Let's Learn a Shortcut

To divide by 5:

To divide a number by 5 we multiply it by $\frac{1}{5}$. But $\frac{1}{5} = \frac{2}{10}$, so we can multiply by 2 and then divide by 10 (both of these are easy to do mentally).

Example:

$$41 \div 5 = 41 \times \tfrac{1}{5} = 41 \times \tfrac{2}{10}$$

Double: 82

Take $\frac{1}{10}$: 8.2

Try these:

1. $\frac{1}{5} \times 37$
2. $82 \div 5$
3. $\frac{1}{5} \times 212$
4. $318 \div 5$
5. $\frac{1}{5} \times 73$
6. $^-87 \div 5$
7. $\frac{1}{5} \times {}^-124$
8. $^-318 \div {}^-5$

Make up your own shortcut for dividing by 50 and practice it.

ADDITION OF POLYNOMIALS

Now we know what polynomials are. We also know how to talk about them. The set for our new mathematical system will be the set of all polynomials in one variable with rational number coefficients.

How shall we define addition of polynomials? Let's remember how we shall want to use them. If we have a *sum* of polynomials, like this:

$$(x^2 + 2) + (2x^2 + 3)$$

it will name a number for each replacement. We will want closure, so we will want a *polynomial* for the sum. It will name some number for every replacement. We shall want the new expression to be *equivalent* to the one above. That way they will name the same number for every replacement. Then we can use either expression whenever we wish.

If we base our definition on the distributive property, we know we will get equivalent expressions.

8.4 Let's Explore

1. Let's think about the polynomials $x^2 + 2$ and $2x^2 + 3$.
   a. Write them, and put a plus sign between them.
   b. Look at the expression you just wrote. Combine similar terms.
   c. Is the expression you wrote for part b a polynomial?
   d. Will it name the same number as the expression of part a, for *every* rational number replacement? Why?

2. Let's think about the polynomials ${}^{-}3x^3 + 2x + {}^{-}4$ and $4x^3 + 3x^2 + 2$.
   a. Write them, and put a plus sign between them.
   b. Look at the expression you just wrote. Combine similar terms and arrange in descending order.
   c. Is the expression you just wrote a polynomial?
   d. Will it name the same number as the expression of part a, for *every* rational number replacement? Why?

3. Let's think about the polynomials $\frac{2}{3}x^4 + 3x^2 + {}^{-}2x + \frac{1}{2}$ and ${}^{-}\frac{1}{3}x^4 + 5x^3 + {}^{-}3x^2 + 3x + {}^{-}\frac{1}{2}$.
   a. Write them, and put a plus sign between them.
   b. Look at the expression you just wrote. Combine similar terms and arrange in descending order.
   c. Is the expression you just wrote a polynomial?
   d. Will it name the same number as the expression of part a, for *every* rational number replacement? Why?

Perhaps we can now make a useful definition of addition for polynomials. We can write them, put a plus sign between, and then combine similar terms. Let's make this definition.

The **sum** *of two polynomials is a polynomial obtained by writing a plus sign between them and then combining similar terms.*

This definition tells us what a *sum* is. It also tells us one way to find it. This doesn't mean that we have to find it that way. Later we shall find a different way to find sums of polynomials.

EXERCISES

In exercises 1-10, find the sum of each pair of polynomials.

Example:

$$(4x^2 + {}^{-}2x + 3) + ({}^{-}x^2 + 5x + {}^{-}5) \equiv$$
$$4x^2 + {}^{-}x^2 + {}^{-}2x + 5x + 3 + {}^{-}5 \equiv$$
$$3x^2 + 3x + {}^{-}2$$

1. $3x^2 + 2x + {}^{-}2$ and ${}^{-}2x^2 + 5x + 5$
2. ${}^{-}4x^5 + 3x^3 + 4$ and $7x^4 + 2x^2$
3. $31x^4 + x^2 + 2x + {}^{-}1$ and ${}^{-}7x^4 + 5x^3 + {}^{-}2x + 2$
4. $17x^3 + {}^{-}x^2 + 3x + 4$ and ${}^{-}15x^3 + x^2 + {}^{-}3x + {}^{-}4$
5. $5x^2 + {}^{-}x + 3$ and ${}^{-}5x^2 + x + {}^{-}3$
6. $3x$ and $\frac{14}{3}x$
7. $\frac{18}{5}$ and ${}^{-}7$
8. ${}^{-}12x^6 + 17x^4 + 5x^2 + 2$ and $10x^6 + 6x^5 + 7x^2$
9. $9x^7 + 2x^3 + {}^{-}x + {}^{-}3$ and $4x^7 + {}^{-}2x^3 + 5x + 3$
10. $13x^3 + 4x^2 + 5$ and $2x^3 + 5x^2 + {}^{-}4x + 2$
11. a. Write an expanded numeral for 43542_b.
    b. Write an expanded numeral for 112103_b.
    c. Your answers to parts a and b should be polynomials with the variable 'b'. Add them.
    d. Use your answer to part c to write a base b numeral for the sum of 43542_b and 112103_b.

Polynomials, as we defined them, always have plus signs between the terms. Sometimes expressions which have some minus signs in them are useful. We can still add, in about the same way, if we remember that to subtract is to add an inverse.

Example:

Add $3x^2 - 2x + 2$ and $5x^3 - 2x^2 + 3x - 4$.

$$(3x^2 - 2x + 2) + (5x^3 - 2x^2 + 3x - 4) \equiv$$
$$5x^3 + (3 - 2)x^2 + ({}^{-}2 + 3)x + (2 - 4) \equiv$$
$$5x^3 + x^2 + x - 2$$

EXERCISES

Add each pair of expressions.

1. $4x^2 - 5x + 3$ and ${}^{-}2x^2 + 2x - 4$
2. $3x^3 - 4x^2 - 5x + 3$ and $5x^3 + 2x^2 - 3x - 4$
3. ${}^{-}5x^3 + 7x^2 - x + 1$ and $8x^3 - 5x^2 + 3x - 3$
4. $8x^4 - 2x^3 + 5x^2 + 5$ and ${}^{-}8x^4 - 2x^3 - 3x^2 + 5$
5. $17x^6 + 7x^4 - 4x^3 + 6$ and ${}^{-}15x^6 - 10x^4 - 2x^3 + 5x^2 - 8$

PROPERTIES OF ADDITION FOR POLYNOMIALS

We have already seen that the set of polynomials is closed under addition. We made the definition so that it would be. Now let's think about some other properties.

8.5 Let's Explore

1. Add:
   a. $(2x^3 + {}^{-}4x^2 + 7) + (x^3 + 3x^2 + {}^{-}x + {}^{-}2)$
   b. $(x^3 + 3x^2 + {}^{-}x + {}^{-}2) + (2x^3 + {}^{-}4x^2 + 7)$
   c. $(3x^5 + {}^{-}4x + 7) + ({}^{-}2x^3 + 5x^2 + {}^{-}3)$
   d. $({}^{-}2x^3 + 5x^2 + {}^{-}3) + (3x^5 + {}^{-}4x + 7)$
   e. $({}^{-}6x^6 + 4x^4 + x) + (x^6 + {}^{-}4x^4 + 6x + 3)$
   f. $(x^6 + {}^{-}4x^4 + 6x + 3) + ({}^{-}6x^6 + 4x^4 + x)$
2. What property does exercise 1 suggest?
3. Can you prove that polynomials have that property?
4. Add:
   a. $(3x + 4) + [(3x^2 + {}^{-}2) + ({}^{-}x^2 + x + {}^{-}1)]$

b. $[(3x + 4) + (3x^2 + {}^{-}2)] + ({}^{-}x^2 + x + {}^{-}1)$
c. $({}^{-}4x^2 + {}^{-}x + 2) + [(2x^2 + 3x) + (2x^2 + 4)]$
d. $[({}^{-}4x^2 + {}^{-}x + 2) + (2x^2 + 3x)] + (2x^2 + 4)$
e. $(5x + {}^{-}5) + [(x^2 + x + 5) + (x^3 + 2x)]$
f. $[(5x + {}^{-}5) + (x^2 + x + 5)] + (x^3 + 2x)$

5. What property does exercise 4 suggest?

6. See if you can prove that polynomials have that property.

We have now seen that polynomials have the commutative and associative properties of addition.

Polynomials in one variable, with rational number coefficients have the commutative and associative properties of addition.

Is there an additive identity in the system of polynomials? Remember that numerals by themselves are polynomials. Then '0' is a polynomial. If we add it to any other polynomial we will get that other polynomial.

The polynomial '0' is the additive identity in the system of polynomials.

The next question we will want to ask is: Which polynomials have additive inverses?

8.6 Let's Explore

1. Add these polynomials:

a. $0 + 0$
b. $2 + {}^{-}2$
c. $2x + {}^{-}2x$
d. ${}^{-}5x^2 + 5x^2$
e. $(4x^3 + 2) + ({}^{-}4x^3 + {}^{-}2)$
f. $(5x^4 + 2x^3 + 5x + 7) + ({}^{-}5x^4 + {}^{-}2x^3 + {}^{-}5x + {}^{-}7)$

g. $(4x^3 + x^2 + x) + ({}^-4x^3 + {}^-x^2 + {}^-x)$
h. $(7x^3 + {}^-2x^2 + x + {}^-3) + ({}^-7x^3 + 2x^2 + {}^-x + 3)$
i. $({}^-13x^6 + 2x^4 + {}^-3x^2 + x + {}^-5) + (13x^6 + {}^-2x^4 + 3x^2 + {}^-x + 5)$
j. $(12x^4 + {}^-3x^2 + 4x) + ({}^-4x^4 + 3x^2 + {}^-4x)$

2. What have you discovered about additive inverses of polynomials?

You have seen that some polynomials have additive inverses. Did you prove that *all* polynomials have additive inverses? We can see that they do. To find the additive inverse of a polynomial, we replace every coefficient by its additive inverse.

Every polynomial has an additive inverse. To find the additive inverse of a polynomial, replace every coefficient by its additive inverse.

SYMBOLS FOR ADDITIVE INVERSES

The additive inverse of $3x^2 + {}^-4x + 3$ is the polynomial we get by replacing every coefficient by its additive inverse. So the inverse is ${}^-3x^2 + 4x + {}^-3$. We can also use the usual symbol for additive inverses. We can represent the additive inverse of $3x^2 + {}^-4x + 3$ like this

$$^-(3x^2 + {}^-4x + 3).$$

Then we know that the following sentence is true:

$$^-(3x^2 + {}^-4x + 3) \equiv {}^-3x^2 + 4x + {}^-3.$$

This sentence tells us that the additive inverse of $3x^2 + {}^-4x + 3$ is ${}^-3x^2 + 4x + {}^-3$.

EXERCISES

1. Write the additive inverse of each polynomial.
   a. $5x^2 + {}^-x + 4$
   b. ${}^-5x^4 + 7x^3 + {}^-3x + 7$
   c. $14x^5 + {}^-5x^3 + {}^-6x + {}^-4$
   d. ${}^-x^7 + {}^-2x^5 + {}^-17x^3 + {}^-4x + {}^-6$
   e. $13x^8 + 5x^6 + x^3 + 4x + 9$

2. For each expression, write the polynomial equivalent to it:

Example: $^-(4x^3 + {}^-6x + 3) \equiv {}^-4x^3 + 6x + {}^-3$

a. ${}^-(5x^4 + 3x^2 + 7x + 5)$
b. ${}^-({}^-6x^4 + {}^-7x^3 + {}^-13x^2 + {}^-x)$
c. ${}^-(14x^{10} + {}^-\frac{1}{2}x^5 + 5x^3 + {}^-x^2 + 3x + {}^-5)$
d. ${}^-(x^6 + 5x^4 + 3x^2 + {}^-x + {}^-10)$
e. ${}^-(x^4 + x^3 + x^2 + x)$

SUBTRACTION OF POLYNOMIALS

To subtract one polynomial from another, we will of course add the inverse of the subtrahend to the minuend.

Example:

$$(4x^5 + x^3 + {}^-2x^2 + 4) - (2x^5 + x^4 + {}^-4x^3 + {}^-3x^2) =$$
$$(4x^5 + x^3 + {}^-2x^2 + 4) + {}^-(2x^5 + x^4 + {}^-4x^3 + {}^-3x^2) =$$
$$(4x^5 + x^3 + {}^-2x^2 + 4) + ({}^-2x^5 + {}^-x^4 + 4x^3 + 3x^2) =$$
$$2x^5 + {}^-x^4 + 5x^3 + x^2 + 4$$

EXERCISES

Subtract:

1. $(5x^2 + 4) - (2x^2 + 1)$
2. $(7x^3 + 2x + 4) - (5x^3 + 4)$
3. $(3x^4 + 6x^2 + 7) - (6x^4 + 2x^2 + 3x + 3)$
4. $(5x^2 + 4x) - (10x^2 + {}^-2x)$
5. $({}^-3x^2 + 5x + {}^-4) - ({}^-4x^2 + 11x + {}^-2)$
6. $(6x^2 + {}^-5x + 2) - (6x^2 + {}^-5x + 2)$
7. $(x^2 + 7x + 3) - ({}^-2x^2 + 5x + 5)$
8. $(4x^2 + 6) - (3x^6 + {}^-2x^4 + 5x^2 + {}^-6)$
9. $(3x^5 + {}^-4x^3 + 12) - ({}^-3x^5 + 4x^3 + {}^-12)$
10. $(14x^4 + {}^-6x^2 + {}^-5) - (4x^4 + {}^-6x^3 + 3x^2 + 8x + 6)$

Perhaps you discovered that you don't have to write so many steps in doing a subtraction problem. You can *think* through finding the inverse of the subtrahend without writing every step. Here is an example:

$$(4x^5 + x^3 + {}^-2x) - ({}^-2x^5 + 5x^3 + 6)$$

We know we must add the additive inverse of the subtrahend. We also know that we get the additive inverse by taking the inverse of each term. We can do this mentally.

We think: The inverse of ${}^{-}2x^5$ is $2x^5$. This combines with $4x^5$ to give $6x^5$.
We write the $6x^5$.

We think: The additive inverse of $5x^3$ is ${}^{-}5x^3$. This combines with x^3 to give ${}^{-}4x^3$.
We write the ${}^{-}4x^3$.

We think: There is no 'x' term in the subtrahend, so in our answer there will be ${}^{-}2x$.
We write the ${}^{-}2x$.

We think: The additive inverse of 6 is ${}^{-}6$. This does not combine with any term of the minuend.
We write ${}^{-}6$.

Then we will have written:

$$6x^5 + {}^{-}4x^3 + {}^{-}2x + {}^{-}6$$

This is the difference of the two polynomials. By thinking this way, we can write the difference without using several steps. See if you can do the following exercises that way.

EXERCISES

Subtract:

1. $({}^{-}6x^4 + 3x^2 + 6) - (2x^4 + 5x^3 + {}^{-}5x^2 + 7)$
2. $(7x^3 + 3x + 2) - ({}^{-}4x^3 + 6)$
3. $({}^{-}2x + 5) - (2x^2 + 3)$
4. $(6x^3 + 5x) - (8x^3 + {}^{-}3x)$
5. $(4x^2 + 5x + {}^{-}2) - (4x^2 + 5x + {}^{-}2)$
6. $({}^{-}2x^2 + 4x + {}^{-}3) - ({}^{-}4x^2 + 6x + {}^{-}5)$
7. $({}^{-}x^2 + 5x + 2) - ({}^{-}3x^2 + {}^{-}x + 3)$
8. $(3x^2 + {}^{-}2) - (2x^6 + {}^{-}5x^4 + 5x^2 + {}^{-}5)$
9. $({}^{-}2x^3 + 3x^2 + {}^{-}5) - (x^4 + 5x^3 + {}^{-}2x^2 + 5x)$
10. $(3x^2 + {}^{-}2x + 5) - (5x^2 + 2x + 5)$
11. $(4x^3 + 2x + 5) - ({}^{-}5x^3 + 4)$
12. $({}^{-}5x^4 + 2x^2 + 7) - (4x^4 + 3x^3 + {}^{-}4x^2 + 7)$

13. $(8x^5 + 2x + 5) - ({}^-2x^5 + 6)$

14. $({}^-3x^2 + 5x + {}^-5) - ({}^-2x^2 + 6x + {}^-3)$

15. $(5x^2 + {}^-7x + {}^-10) - ({}^-3x^2 + 7x + {}^-10)$

16. $(\frac{3}{2}x^3 + {}^-\frac{1}{2}x^2 + .3) - (\frac{1}{2}x^3 + \frac{1}{2}x^2 + \frac{4}{3}x + .2)$

17. $({}^-\frac{1}{3}x^2 + {}^-\frac{4}{5}x + 17) - ({}^-\frac{1}{3}x^2 + \frac{2}{5}x + 15)$

18. $({}^-x^2 + 3x + {}^-5) - ({}^-2x^2 + {}^-x + 4)$

19. $(\frac{3}{2}x^5 + 4x^2 + 5) - ({}^-x^4 + 2x^3 + {}^-2x)$

20. $(\frac{75}{12}x^4 + {}^-\frac{10}{11}x^2 + \frac{14}{9}) - ({}^-\frac{5}{12}x^4 + \frac{2}{3}x^3 + \frac{1}{11}x^2)$

Adding and Subtracting in Columns

When we add polynomials we can write a plus sign between them and then combine similar terms. We do not have to write them that way. Sometimes it is better to write one polynomial under the other. If we have more than two polynomials to add it is usually easier to write them one under the other. When we do this, we write similar terms in a column, leaving spaces for missing terms.

Example:

Add $4x^5 + {}^-2x^3 + 6x^2 + 3$, $5x^4 + {}^-7x^2 + 6$, and $3x^6 + {}^-5x^5 + x^2 + 5$.

Arrange the polynomials so that similar terms are in columns:

$$\begin{array}{ccccccccccc}
 & & 4x^5 & & & + & {}^-2x^3 & + & 6x^2 & + & 3 \\
 & & & & 5x^4 & & & + & {}^-7x^2 & + & 6 \\
3x^6 & + & {}^-5x^5 & & & & & + & x^2 & + & 5 \\
\hline
3x^6 & + & {}^-x^5 & + & 5x^4 & + & {}^-2x^3 & & & + & 14
\end{array}$$

EXERCISES

Add: (Use columns)

1.
$$\begin{array}{ccccccccc}
 & & {}^-2x^3 & + & 5x^2 & + & {}^-2x & + & 4\text{, and} \\
x^4 & & & + & 6x^2 & + & 7x & + & {}^-10\text{, and} \\
{}^-3x^4 & + & 6x^3 & + & x^2 & & & + & {}^-2. \\
\hline
\end{array}$$

2. ${}^-3x^3 + 5x + 2$, and $x^3 + x^2 + 5$, and $x^3 + {}^-2x + {}^-4$, and $2x^2 + {}^-3x + 2$.

3. $2x^5 + {}^-5x^3 + 2x + {}^-4$, and $3x^5 + 4x^2 + {}^-5x + 7$, and ${}^-4x^5 + 6x^3 + 7x$, and $5x^2 + x + 1$.

4. $7x^4 + 2x^3 + 5$, and $4x^3 + {}^{-}6x^2 + 5x$, and ${}^{-}3x^4 + 7x + 6$, and ${}^{-}2x^2 + 5x + 7$, and ${}^{-}x^4 + {}^{-}x^2 + {}^{-}2$.

5. ${}^{-}x^3 + {}^{-}x^2 + {}^{-}x$, and $x^4 + 3x^3 + x + 1$, and $4x^2 + {}^{-}x + {}^{-}1$, and $3x^3 + 2x^2 + 2$.

6. $3x^2 - 4x + 2$, and $4x^3 - 2x^2 + 5x - 5$, and ${}^{-}x^2 + 3x - 3$ and $2x^3 - x^2 + 2x - 1$.

7. ${}^{-}6x^5 + 3x^2 - x + 3$, and $3x^3 - 5x^2 - 5$, and $4x^4 - 2x^2 + x + 3$, and $2x^3 - 2x^2 + 3x + 4$.

8. ${}^{-}x^3 - x^2 - x - 1$, and $2x^4 - 3x^2 + 4x - 2$, and ${}^{-}3x^3 - 2x^2 - 2x - 2$, and $5x^3 - x^2 - x - 3$.

9. $3x^5 - 2x^3 + 4x$, and $5x^5 + 2x^4 + 5x^3 + 2x$, and ${}^{-}4x^4 + 3x^2 - 5$, and ${}^{-}x^5 + 5x^2 + 12$.

10. $5x^3 + {}^{-}2x^2 + 12$, and ${}^{-}x^3 + 8x^2 - 5x + 2$, and ${}^{-}3x^3 + 7x - 2$, and $4x^2 - 5x + 7$.

11. $3y^3 + {}^{-}2y^2 - 4$, and ${}^{-}2y^4 + 5y^3 + 14y^2 - 5$, and $7y^3 - 5y^2 + 3$, and ${}^{-}12y^4 + 7y^3 - 5$.

12. $16z^2 - 2z^3 + 4z - 10$, and ${}^{-}4z^5 + 7z^4 - 5z^3$, and ${}^{-}8z^4 + 5z^3 + 7z - 2$, and $21z^3 + 14z^2 - 5$.

13. $1.4x^3 + 3.1x^2 + 14$, and $5.7x^3 - 4.3x^2 + 5.32x$, and ${}^{-}7.3x^3 - 2.5x^2 + 6.5$ and $3.8x^2 - 4.2x - 1.1$.

14. $16y^4 - 7y^3 + 14y + 5$, and ${}^{-}5y^4 + 3y^2 - 7y - 2$, and ${}^{-}7y^4 + 8y^3 - 6y^2$, and $18y^2 - 17$.

15. $35x^3 + x^2 - 6$, and ${}^{-}7x^3 + 4x + 3$, and $2x^3 - 5x^2 + 3x + 4$, and $4x^3 + 7x - 6$.

When we subtract using columns we write one polynomial under the other. The polynomial to be subtracted is written underneath the other one. Remember that we will add the inverse of the subtrahend. We can find the additive inverse by changing each coefficient to its additive inverse. This can be done mentally. Or, if it helps, you can write plus or minus signs like those shown in color in the following example:

Example: Subtract $3x^2 + {}^{-}5x + {}^{-}3$ from $5x^2 + {}^{-}3x + 6$.

$$\begin{array}{crcrcr} & 5x^2 & + & {}^{-}3x & + & 6 \\ - & & & + & & + \\ & 3x^2 & + & {}^{-}5x & + & {}^{-}3 \\ \hline & 2x^2 & + & 2x & + & 9 \end{array}$$

EXERCISES

Subtract the second polynomial from the first in each exercise. (Use columns.)

1. $4x^3 + 2x^2 + {}^{-}2x + {}^{-}3, \quad 2x^3 + {}^{-}3x^2 + 2.$
2. ${}^{-}2x^5 + 2x^4 + {}^{-}3x^3 + 2x + {}^{-}5, \quad {}^{-}x^5 + 4x^3 + {}^{-}5x + 4.$
3. $4x^4 + 6x^3 + 5x + 12, \quad 6x^4 + {}^{-}2x^3 + 17.$
4. $3x^3 + 5x^2 + 2x + {}^{-}7, \quad {}^{-}5x^3 + 3x^2 + {}^{-}x.$
5. ${}^{-}4x^4 + x^3 + 3x^2 + 6, \quad 4x^4 + {}^{-}6x^3 + x + {}^{-}5.$
6. ${}^{-}2x^3 + 2x^2 - 5x + 3, \quad 6x^3 - 4x^2 + x - 5.$
7. ${}^{-}2x^4 + 2x^3 - x^2 + 5, \quad {}^{-}3x^4 - 4x^3 - x^2 - 2.$
8. $2x^3 + x^2 - 6x + 2, \quad x^5 + 4x^3 - 2x^2 - 4x.$
9. ${}^{-}5x^4 + 5x^3 - 3x^2 - 7, \quad 4x^4 - 5x^3 + x^2 + 2.$
10. $6y^5 + 5y^4 - 7y^2 + 7, \quad {}^{-}3y^5 + 4y^3 - 2y - 4.$
11. $3z^4 + 5z^3 - 2z^2 + 15, \quad 7z^4 - 2z^2 + 15.$
12. $14x^3 - 2x^2 + 6, \quad {}^{-}14x^3 + 2x^2 - 6.$
13. ${}^{-}4.1x^2 + 6.2x - 8.5, \quad 3.2x^2 - 5.8x + 7.$
14. $13x^7 + 6x^6 + x^5 + 12, \quad 14x^7 - 6x^5 + x^2 - 12.$
15. $12x^3 + {}^{-}4x^2 + 5x - {}^{-}3, \quad 12x^3 - 4x^2 + 5x + 3.$

Let's Estimate

You have estimated when computing. You have also made estimates of measures. This is a very useful thing to be able to do. It takes a lot of practice, however.

Try these:

1. Estimate the area of the floor of your classroom.
2. Estimate the number of books in your school library.
3. Estimate your walking speed.
4. Estimate the length and width of this page.
5. Estimate the number of square feet of cloth in a window shade.

Find some things whose measures you can estimate. There are many of them around you all the time. If you estimate and can then make an actual measurement, you will know how close your estimate was. This is a good way to improve your estimating ability.

MULTIPLICATION OF POLYNOMIALS

We wish to define multiplication of polynomials so that we have closure. The product of two polynomials should be a polynomial. If we have a product of polynomials, like this

$$(2x^2 + 3)(x + 3)$$

it will name some number for each replacement. The new polynomial we find for the product will also name numbers for each replacement. We want the new expression to be *equivalent* to the one above. Then they will name the same number for every replacement.

8.7 Let's Explore

1. Let's think about the polynomials $3x$ and $5x$.
   a. Write them, with parentheses around them to show multiplication.
   b. Is the expression $(3 \cdot 5)(x \cdot x)$ equivalent to your answer to part a? Why?
   c. Find a polynomial equivalent to $(3 \cdot 5)(x \cdot x)$.
2. Let's think about the product $(^-4x^2)(2x)$.
   a. Is it equivalent to $(^-4 \cdot 2)(x^2 \cdot x)$? Why?
   b. Write a polynomial equivalent to $(^-4 \cdot 2)(x^2 \cdot x)$.
3. Let's think about the product $(5x^3)(^-2)$.
   a. Is it equivalent to $(5 \cdot {}^-2) \cdot x^3$? Why?
   b. What polynomial is equivalent to $(5x^3)(^-2)$?
4. Do you have an idea about how to define multiplication of two monomials? Can you write out your idea? Try.
5. Use your idea on each of these products. In each case, see if you get a polynomial. Also check whether your answer is equivalent to the given expression.
   a. $(6x)(2x)$
   b. $(^-5x)(^-3x)$
   c. $(^-\frac{3}{2}x^2)(4x)$
   d. $(7x^5)(2x^2)$
   e. $(^-8x^3)(3x^2)$
   f. $(3)(^-2)$
   g. $(3x^4)(0)$
   h. $(x)(x)$
   i. $(^-x)(3x)$
   j. $(^-x^2)(^-2x^2)$
   k. $(^-1)(^-x)$
   l. $(^-x^2)(^-x^2)$

You have probably discovered a good way to define multiplication of two monomials. You multiply their coefficients. Then you use properties of exponents so that you have just one variable symbol.

Example:

$$\begin{aligned}(2x^3)({}^{-}3x^2) &\equiv (2 \cdot {}^{-}3)(x^3 \cdot x^2)\\ &\equiv {}^{-}6x^5\end{aligned}$$

To multiply two monomials $\boldsymbol{ax^m}$ *and* $\boldsymbol{bx^n}$, *multiply the coefficients, and then use properties of exponents. The result is the polynomial:*

$$\boldsymbol{abx^{m+n}}$$

EXERCISES

Multiply these monomials:

1. $3x$ and ${}^{-}5$
2. ${}^{-}2$ and 4
3. ${}^{-}x$ and x
4. x and ${}^{-}1$
5. ${}^{-}x$ and ${}^{-}x$
6. ${}^{-}x$ and ${}^{-}1$
7. ${}^{-}x^2$ and x^3
8. ${}^{-}x^3$ and ${}^{-}x^4$
9. $3x^5$ and $4x^2$
10. $5x^3$ and x
11. $4x^5$ and ${}^{-}2x^6$
12. ${}^{-}5x^3$ and ${}^{-}2x^3$
13. ${}^{-}7x^4$ and ${}^{-}x$
14. ${}^{-}4x^2$ and ${}^{-}5$
15. $7x^5$ and 0
16. ${}^{-}385x^7$ and 0
17. ${}^{-}x^3$ and $5x^6$
18. ${}^{-}6x^2$ and ${}^{-}x^2$

Now we want to define multiplication for polynomials other than monomials. We want to define it so that expressions will be equivalent. So we shall base the definition on the distributive property.

8.8 Let's Explore

1. Let's think about the polynomials $4x + 3$ and $2x$.
   a. Write them, with parentheses around them, to show multiplication.
   b. Use the distributive property to write an equivalent expression.

c. Simplify your answer to part b.
d. Is your answer to c a polynomial?

2. Let's think about the polynomials $x + 3$ and $x + 2$.
   a. Write them, with parentheses around them, to show multiplication.
   b. Use the distributive property to write an equivalent expression, like this:
   $$(x + 3)(\quad) + (x + 3)(\quad)$$
   c. Look at the first part of the expression in part b. Use the distributive property to write an expression equivalent to it. (Your answer should be a polynomial.)
   d. Look at the second part of the expression in part b. Use the distributive property to write an expression equivalent to it. (Your answer should be a polynomial.)
   e. Replace the parts of the expression in part b by your answers to c and d.
   f. Look at your answer to part e. Combine similar terms and arrange in descending order.
   g. Is your answer to f a polynomial?

3. Let's think about the product of polynomials $(2x + 3)(x + {}^{-}2)$.
   a. Use the distributive property to write an equivalent expression, like this:
   $$(2x + 3)(\quad) + (2x + 3)(\quad)$$
   b. Use the distributive property again, with the first part of your answer to part a. (Your answer should be a polynomial.)
   c. Do the same thing for the second part of your answer to part a.
   d. Replace the parts of a by your answers to b and c.
   e. Combine similar terms and arrange in descending order.
   f. Is your answer to e a polynomial?

4. Let's think about the product $(x^2 + 2x + {}^{-}1)(x + 3)$.
   a. Use the distributive property to write an equivalent expression, like this:
   $$(x^2 + 2x + {}^{-}1)(\quad) + (x^2 + 2x + {}^{-}1)(\quad)$$
   b. Use the distributive property again, with the first part of a. (Your answer should be a polynomial.)
   c. Do the same thing for the second part of a.
   d. Replace the parts of a by your answers to b and c.
   e. Combine similar terms and arrange in descending order.
   f. Is your answer to e a polynomial?

5. Do you now have an idea about defining multiplication of polynomials? Can you find a good way to say it?

Did you notice that parentheses are sometimes necessary? If we want to show a product of two binomials, then we need parentheses around both of them, like this:

$$(3x^4 + 2)(x + {}^{-}3)$$

If we left them off, we would have

$$3x^4 + 2x + {}^{-}3,$$

and this does not mean the same thing. For a binomial and a monomial we need parentheses, like this:

$$5x^2(3x + 4)$$

For two monomials we do not need parentheses. For example,

$$(5x^4)(3x) \text{ and } 5x^4 3x$$

mean the same thing. (Of course we can simplify $5x^4 3x$ to $15x^5$.) When we need parentheses we will write them. If we do not need them we usually do not write them. It is not wrong to write them when they aren't needed, but it is wrong to leave them off when they are needed.

Let's make sure we understand how products of polynomials are to be defined. In the exploratory exercises you used the distributive property. Sometimes you needed to use it several times. Perhaps you noticed that finally you had multiplied each term of one polynomial by every term of the other one. When you use the distributive property, this is always what happens. We can make our definition accordingly.

To multiply two polynomials (where at least one of them is not a monomial), multiply each term of one polynomial by every term of the other, then add the results.

This definition tells us how to find the product of two polynomials. It does not, of course, say what we must write. As in addition, there are several good ways of writing symbols to find a product. One way is to write one polynomial under the other. Then multiply each term of the top one by every term of the bottom one. Then we add the results, combining similar terms. To help in adding, we can write similar terms in columns.

Example:

Multiply $3x^2 + {}^{-}2x + 3$ by $x + 2$.

$$
\begin{array}{rrrrl}
3x^2 + & {}^{-}2x & + 3 & & \\
x + 2 & & & & \\
\hline
3x^3 + & {}^{-}2x^2 + & 3x & & \text{(Multiplying each term by } x\text{)} \\
 & 6x^2 + & {}^{-}4x + & 6 & \text{(Multiplying each term by 2)} \\
\hline
3x^3 + & 4x^2 + & {}^{-}1x + & 6 & \text{(Adding)}
\end{array}
$$

Example:

Multiply $x^3 + {}^{-}3x + 4$ by ${}^{-}2x^2 + {}^{-}3$.

$$
\begin{array}{lrrrl}
x^3 + {}^{-}3x + 4 & & & & \\
{}^{-}2x^2 + {}^{-}3 & & & & \\
\hline
{}^{-}2x^5 & + \; 6x^3 + & {}^{-}8x^2 & & \text{(Multiplying by } {}^{-}2x^2\text{)} \\
 & - \; 3x^3 & & + \; 9x + {}^{-}12 & \text{(Multiplying by } {}^{-}3\text{)} \\
\hline
{}^{-}2x^5 & + \; 3x^3 + & {}^{-}8x^2 & + \; 9x + {}^{-}12 & \text{(Adding)}
\end{array}
$$

Notice that a space has been left for "missing terms." You should do this, too. It will help you in adding.

EXERCISES

1. Multiply these polynomials. Write your work like the examples above whenever you find it easier.
   a. $4x^2 + 2x + {}^{-}5$ and $2x$
   b. ${}^{-}4x$ and ${}^{-}x^3 + 4x + {}^{-}5$
   c. $x + 2$ and $x + 3$
   d. $3x + 5$ and $2x + 3$
   e. ${}^{-}2x + 3$ and $x + 1$
   f. $5x + {}^{-}2$ and $2x^2 + 2$
   g. $x^4 + {}^{-}2x^3 + 3$ and $x + 2$
   h. ${}^{-}x^2 + x$ and ${}^{-}2x^5 + 3x^2 + {}^{-}1$
   i. ${}^{-}3x^2 + 2$ and $4x^4 + {}^{-}7x^2 + 3$
   j. $x^2 + 2x + 2$ and $2x^2 + {}^{-}x + 1$

2. a. Write expanded numerals (base b) for 314_b and 21_b.
   b. Your answers to a should be polynomials. Multiply them.
   c. Assuming that $b > 9$, write the base b numeral for the product.

3. a. Write expanded numerals (base b) for 103_b and 212_b.
   b. Your answers to a should be polynomials. Multiply them.
   c. Assuming that $b > 8$, write the base b numeral for the product.

4. a. Write an expanded numeral (base b) for 31246_b.
   b. Your answer to a should be a polynomial. Multiply it by b.
   c. Now write the base b numeral for your answer to part b.

5. a. Write an expanded numeral (base b) for 40532_b.
   b. Your answer to a should be a polynomial. Multiply it by b^3.
   c. Now write the base b numeral for your answer to part b.

6. a. Write an expanded numeral (base ten) for 31246_{10}.
   b. Multiply by 10^2. Do it as you multiply polynomials.
   c. Now write the decimal numeral for your answer to part b.

7. Which of these sentences about polynomials are true?
   a. $x(x^2 + 3) \equiv x^3 + 3x$
   b. ${}^-x^2(x + {}^-5) \equiv {}^-x^3 + 5x$
   c. $(x + 2)3x \equiv 3x^2 + 5x$
   d. $(3x^5)({}^-x^2) \equiv {}^-3x^3$
   e. $(3x^5)({}^-x^2) \equiv {}^-3x^{10}$
   f. $(3x^5)({}^-x^2) \equiv 3x^7$
   g. $(x + 2)(x + 3) \equiv x^2 + 5x + 5$
   h. $(x + 5)(x + 4) \equiv x^2 + 9x + 20$
   i. $(x + 3)(x - 2) \equiv x^2 + x - 6$
   j. $(x - 3)(x - 3) \equiv x^2 - 9$
   k. $(x - 3)(x + 3) \equiv x^2 - 9$

*PROPERTIES OF MULTIPLICATION FOR POLYNOMIALS

The set of polynomials with one variable is closed under multiplication. We made the definition so that this would be true. What about other properties? Is multiplication associative? Commutative? Do polynomials have the distributive property of multiplication over addition? Is there a multiplicative identity? If so, which polynomials have multiplicative inverses?

8.9 Let's Explore

1. Multiply:
   a. $(x + 2)(2x + 3)$
   b. $(2x + 3)(x + 2)$
   c. $(x + {}^-5)(x + 4)$
   d. $(x + 4)(x + {}^-5)$
   e. $(x^2 + 2x + 1)(3x + 4)$
   f. $(3x + 4)(x^2 + 2x + 1)$

2. What property does exercise 1 suggest? Can you prove that polynomials have that property?

3. Multiply:
   a. $[(x + 2)(2x + 3)](x + 1)$
   b. $(x + 2)[(2x + 3)(x + 1)]$
   c. $(2x + 1)[(x^2 + 4)(x + {}^-1)]$
   d. $[(2x + 1)(x^2 + 4)](x + {}^-1)$
   e. $(x^2 + 2x + 3)[(x + 4)(2x^2 + 3x + 1)]$
   f. $[(x^2 + 2x + 3)(x + 4)](2x^2 + 3x + 1)$.

4. What property does exercise 3 suggest? Can you prove that polynomials have that property?

You have seen that multiplication of polynomials is commutative and associative. Polynomials have these properties because rational numbers have them. There is a multiplicative identity. It is the polynomial 1. There is one property, however, which rational numbers have, but polynomials do not. Not all polynomials have multiplicative inverses. You will learn more about this later, when you study division of polynomials.

SPECIAL PRODUCTS OF POLYNOMIALS

Now we have defined multiplication of polynomials and have practiced doing it. Multiplication of polynomials is important. We shall often need to multiply. It will help us if we study certain polynomial products. If we memorize special ones, we shall be able to work more quickly. Already you have probably noticed something about a product of a monomial and some other polynomial, like:

$$3x(4x^2 + 3x + {}^-2)$$

You can use the distributive property mentally. All you need to write is the answer.

PRODUCTS OF TWO BINOMIALS

When we multiply two binomials, we use the distributive property three times. This amounts to multiplying each term of one binomial by every term of the other. Can we find a way to do this mentally?

8.10 Let's Explore

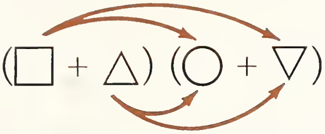
1. Multiply these binomials. Look for some pattern.
   a. $(x^2 + 3)(x + 2)$
   b. $(x^3 + 1)(x + 2)$
   c. $(x^2 + 3)(2x + {}^-2)$
   d. $(3x^2 + 2)(2x + {}^-3)$
2. Do you see a pattern? If so, explain it. If not, make up some more examples.
3. Let's use these symbols to represent the terms of two binomials:

   □, △, ○, ▽

   Explain the pattern by finishing this sentence.

   $(\square + \triangle)(\bigcirc + \triangledown) =$

Did you "discover" that products of two binomials have four terms? Very often they do. (We shall see later that in some cases terms can be combined.) In any case we can always write down four terms. We can explain how to do this mentally, as follows:



We multiply □ by ○ and □ by ▽. Then we multiply △ by ○ and △ by ▽. Then we add the results.

Sometimes when we do this we find that some terms are similar. If they are, then we "combine" them. We do this so that the result will be a polynomial.

Examples:

$$\begin{aligned}(x+2)(x+3) &= x^2+3x+2x+6\\ &= x^2+5x+6\end{aligned}$$

$$\begin{aligned}(x+3)(x-3) &= x^2-3x+3x-9\\ &= x^2-9\end{aligned}$$

You will need some practice in multiplying two binomials mentally. Use the following exercises to see how fast you can do it without making a mistake.

EXERCISES

Multiply mentally. Write only the answer.

1. $(x+4)(x+3)$
2. $(x+2)(x-3)$
3. $(2x+1)(x+3)$
4. $(3x+1)(5x-2)$
5. $(x^2+3)(x^2+x)$
6. $(2x^2-2)(x-4)$
7. $(^{-}x+5)(x^2+3)$
8. $(^{-}2x+1)(4x-2)$
9. $(^{-}x-5)(x+5)$
10. $(6x^2+3)(2x^3+1)$
11. $(x^4+2x)(x-1)$
12. $(2x^4+x^2)(^{-}x^3+x)$
13. $(x^3+5)(x^3-5)$
14. $(5x-4)(^{-}3x+1)$

15. $(9x^3 + {}^{-}4x)(x^2 + 1)$
16. $(5x^2 + {}^{-}3)(5x^2 + 3)$
17. $(3x + 4)(2x - 3)$
18. $(9x^2 + 5)({}^{-}2x + 5)$
19. $(6x + 3)(2x - 1)$
20. $({}^{-}x + 5)(3x - 5)$
21. $(5x + 2)(2x + 3)$
22. $(3x - 1)(2x + 4)$
23. $(3x + 5)(2x + 1)$
24. $(4x - 3)(2x + 5)$
25. $(x + 4)(x - 2)$
26. $(x + 2)(x + 3)$
27. $(x + 5)(x + 4)$
28. $(x - 5)(x - 3)$
29. $(5x - 3)(2x - 4)$
30. $(4x - 1)(3x + 2)$

When two binomials have no terms higher than first degree and both binomials contain the same variable, then some terms always combine. Therefore the polynomial which is the product never has four terms.

Examples:

$$\begin{aligned}(x + 3)(x - 2) &\equiv x^2 - 2x + 3x - 6\\ &\equiv x^2 + x - 6\end{aligned}$$

$$\begin{aligned}(2x + 5)(x + 4) &\equiv 2x^2 + 8x + 5x + 20\\ &\equiv 2x^2 + 13x + 20\end{aligned}$$

Cases like these are still more special. We multiply the first terms of the binomials. This gives us the first term of the product. We multiply the "outside" terms and the "inside" terms and add. These always can be "combined." Their sum is the next term of the product. We multiply the second terms. This gives us the last term of the product. With a little more practice you should be able to multiply very quickly in cases like this.

EXERCISES

Multiply mentally.

1. $(x - 3)(x + 5)$
2. $(x + 3)(x + 5)$
3. $(x - 2)(x - 6)$
4. $(x + 6)(x - 5)$
5. $(x - 1)(x+ 2)$
6. $(x - 3)(x - 2)$
7. $(2x + 3)(x + 4)$
8. $(x + 5)(3x + 1)$

9. $(4x - 2)(x + 2)$
10. $(6x + 4)(x - 5)$
11. $(4x + 1)(2x + 3)$
12. $(5x + 1)(2x + 3)$
13. $(4x - 3)(2x + 1)$
14. $(5x + 2)(3x - 3)$
15. $(5x - 2)(2x - 1)$
16. $(2x - 6)(3x - 2)$
17. $(^{-}2x + 1)(x + 3)$
18. $(3x + 4)(^{-}x - 2)$
19. $(^{-}2x + 5)(^{-}x + 2)$
20. $(^{-}3x - 2)(2x + 4)$

There are other special kinds of products of two binomials. You have just practiced on a kind where the products have three terms. Now we wish to look at another kind.

8.11 Let's Explore

1. Multiply. Look for a pattern.
   a. $(x - 2)(x + 2)$
   b. $(x + 4)(x - 4)$
   c. $(2x - 1)(2x + 1)$
   d. $(3x + 5)(3x - 5)$
   e. $(x^2 + 4)(x^2 - 4)$
   f. $(2x^2 + 1)(2x^2 - 1)$
   g. $(x^2 - x)(x^2 + x)$
   h. $(2x^3 + 5x)(2x^3 - 5x)$
   i. $(2x + 4)(2x + {}^{-}4)$
   j. $(^{-}2x + 3)(^{-}2x + {}^{-}3)$
   k. $(^{-}x + 2)(^{-}x - 2)$
   l. $(x - 2)(2 + x)$
2. What pattern do you see? Can you explain how to multiply quickly in cases like these?
3. Explain the pattern by finishing this sentence:

$$(\square + \bigcirc)(\square - \bigcirc) = \underline{\hspace{6cm}}$$

What is special about the pairs of binomials in the above exercises? Did you notice that the binomials of a pair have one term the same? Did you notice that the other terms are additive inverses of each other? Whenever we multiply a pair of binomials like this we find that the product has just two terms:

$$(\square + \bigcirc)(\square - \bigcirc) = \square^2 - \bigcirc^2$$

Find the second power of each term and write a minus sign between them. Now with some practice, you will be able to do multiplications of this type very quickly indeed.

EXERCISES

Multiply mentally:

1. $(x + 3)(x - 3)$
2. $(2x + 1)(2x - 1)$
3. $(x - \frac{1}{2})(x + \frac{1}{2})$
4. $(3x - 2)(3x + 2)$
5. $(x^2 - x)(x^2 + x)$
6. $(7x + 2)(7x - 2)$
7. $(5x + 3)(5x - 3)$
8. $({}^-x + 1)({}^-x - 1)$
9. $(2x^2 + 1)(2x^2 - 1)$
10. $({}^-2x + 3)({}^-2x - 3)$
11. $(3x^2 - x)(3x^2 + x)$
12. $({}^-x + 2)({}^-x - 2)$
13. $(3x^3 - 2)(3x^3 + 2)$
14. $({}^-x^4 + 5)({}^-x^4 - 5)$
15. $(x + \frac{3}{4})(x - \frac{3}{4})$
16. $(3x^2 - \frac{4}{5})(3x^2 + \frac{4}{5})$
17. $(\frac{2}{3}x - 3)(\frac{2}{3}x + 3)$
18. $(.5x + 2)(.5x - 2)$
19. $(\frac{3}{5}x + .4)(\frac{3}{5}x - .4)$
20. $(.2x^3 - .5)(.2x^3 + .5)$
21. $(\frac{4}{3}x^6 + 3)(\frac{4}{3}x^6 - 3)$
22. $(\frac{3}{8}x^5 - \frac{1}{2})(\frac{3}{8}x^5 + \frac{1}{2})$
23. $(8x^3 + \frac{9}{5})(8x^3 - \frac{9}{5})$
24. $({}^-12x^3 + \frac{8}{3})({}^-12x^3 - \frac{8}{3})$
25. $(\frac{1}{7}x^4 + x)(\frac{1}{7}x^4 - x)$
26. $(3x^5 - 2x^2)(3x^5 + 2x^2)$
27. $(6x^8 - 5x^5)(6x^8 + 5x^5)$
28. $(11x^3 - \frac{1}{2}x^2)(11x^3 + \frac{1}{2}x^2)$
29. $(\frac{4}{11}x^{10} - 3x^2)(\frac{4}{11}x^{10} + 3x^2)$
30. $(20x^3 - .2x)(20x^3 + .2x)$

We wish to study one more special kind of product. This is the kind in which we multiply a binomial by itself, or as we sometimes say, "square a binomial."

8.12 Let's Explore

1. Multiply:
   - a. $(x + 1)(x + 1)$
   - b. $(x - 2)(x - 2)$
   - c. $(2x + 3)(2x + 3)$
   - d. $(x^2 + 5)(x^2 + 5)$
   - e. $(x + 3)(x + 3)$
   - f. $(x - 3)(x - 3)$
   - g. $(x + 5)(x + 5)$
   - h. $(x - 5)(x - 5)$
   - i. $(3x + 2)(3x + 2)$
   - j. $(3x - 2)(3x - 2)$
   - k. $(x + 5)^2$
   - l. $(3x - 5)^2$
   - m. $({}^-2x + 1)^2$
   - n. $(5x^2 + 3)^2$
   - o. $(x^3 - 6)^2$
   - p. $({}^-4x - 7)^2$

2. What pattern do you see? Can you explain how to square a binomial quickly?

3. Explain the pattern by finishing this sentence:

$$(\square + \bigcirc)^2 = \underline{\hspace{8cm}}$$

You should have discovered a quick way to square a binomial. Find the second power of each term. Multiply the two terms and double. Then add. There will be three terms, like this:

$$(\square + \bigcirc)^2 = \square^2 + 2 \cdot \square \cdot \bigcirc + \bigcirc^2$$

or

$$(\square - \bigcirc)^2 = \square^2 - 2 \cdot \square \cdot \bigcirc + \bigcirc^2$$

EXERCISES

Multiply mentally:

1. $(x + 2)(x + 2)$
2. $(3x + 1)(3x + 1)$
3. $(x + 3)^2$
4. $(4x + 1)^2$
5. $(x - 1)^2$
6. $(5x + 4)^2$
7. $({}^-x + 3)^2$
8. $(9x + 3)^2$
9. $({}^-2x - 3)^2$
10. $(8x - 5)^2$
11. $(x^2 + 2)^2$
12. $(2x^3 - x)^2$
13. $({}^-5x^2 + 3x)^2$
14. $(7x^4 - 2x^3)^2$
15. $({}^-4x^3 + 5x)^2$
16. $(5x^5 + 6x^3)^2$
17. $(\frac{1}{2}x^2 + 4)^2$
18. $(\frac{2}{3}y^4 - \frac{1}{2}y)^2$
19. $(.4z^3 - .6z^2)^2$
20. $(\frac{3}{8}x^2 + \frac{3}{4}x)^2$

Now that you have learned how to multiply certain polynomials quickly, suppose we give you some different types mixed together. You should first look to see what type each problem is, and then use your quickest method for that type. Use these exercises to see how fast you can get without making a mistake.

EXERCISES

Multiply mentally:

1. $(3x^2)(2x)$
2. $(^{-}x^4)(2x^3)$
3. $(^{-}5x^2)^2$
4. $(7x + 2)^2$
5. $(^{-}2x)(3x + 4)$
6. $(x - 2)(x - 3)$
7. $(x - 2)(x + 2)$
8. $(2x^2 + 3)(2x^2 - 3)$
9. $(x^2 + 3)^2$
10. $(4x^2)(2x^2 + x + 3)$
11. $(x + \frac{1}{2})(x + \frac{1}{2})$
12. $(x + 5)(x - 3)$
13. $(- x^3)(^{-}2x)$
14. $(^{-}x)(x^3 + 2x^2 - 2)$
15. $(3x^3)^2$
16. $(3x - 5)(3x + 5)$
17. $(x - \frac{1}{2})^2$
18. $(2x + 3)(x - 5)$
19. $(4x - 3)(4x + 3)$
20. $(2x + 5)^2$
21. $(\frac{3}{4}x^3)(^{-}3x)$
22. $(4x + \frac{1}{2})(3x - \frac{1}{2})$
23. $(6x - 2)(6x + 2)$
24. $(\frac{1}{4}x^2 + 2)^2$
25. $(5x - 2)(5x - 2)$
26. $(17x^3)(^{-}2x^2)$
27. $(\frac{1}{2}x + \frac{2}{3})(\frac{1}{2}x - \frac{2}{3})$
28. $(7x + 2)(2x - 7)$
29. $(3x - 7)(3x - 7)$
30. $(3x - 7)(3x + 7)$

Let's Learn a Shortcut

To multiply by 49:

We know that $49 = 50 - 1$, so for any number n, $49 \cdot n = (50 - 1) \cdot n$, and by the distributive property this is the same as $50 \cdot n - 1 \cdot n$. So, to multiply a number by 49 we can multiply it by 50 and then subtract the number.

Example:

$$49 \times 32$$

Multiply by 50 (multiply by 100 and then take half):

$$1600$$

Now subtract 32:

$$1568$$

Try these:

1. 49×46
2. 49×54
3. 49×88
4. $49 \times {}^{-}92$
5. 49×112
6. $49 \times {}^{-}250$
7. 49×316
8. ${}^{-}49 \times {}^{-}824$

Make up your own shortcut for multiplying by 51 and practice it.

FACTORING POLYNOMIALS

In the last chapter we saw that factoring polynomials will help in solving equations. If we know how to factor $x^2 + x - 6$, then we can solve the equation $x^2 + x - 6 = 0$ by the principle of zero products. Factoring is just the reverse of multiplying so in some cases we know how to do it. If we practice we can learn to factor certain polynomials quickly. You will then see how to use your new skill in solving equations. You will learn to factor $x^2 + x - 6$ as follows:

$$x^2 + x - 6 \equiv (x + 3)(x - 2)$$

Then you will know that the equation $x^2 + x - 6 = 0$ has the same solution set as $(x + 3)(x - 2) = 0$, and by the principle of zero products the solution set is $\{2, {}^{-}3\}$.

The polynomials we are studying have rational numbers for coefficients but most of them we use have integers for coefficients. When we study factoring we shall study mostly those with integers for coefficients. This will make it easier.

To learn factoring, we can study our quick methods for multiplying and use them in reverse.

FACTORING MONOMIALS

When we multiply two monomials, we multiply the coefficients and then use the properties of exponents. Let's see how to do this in reverse.

8.13 Let's Explore

1. a. Multiply: $(2x)(3x)$
   b. Factor $6x^2$
2. a. Multiply: $(x)(20x)$
   b. Multiply: $(4x)(5x)$
   c. Multiply: $(2x)(10x)$
   d. Factor $20x^2$ three different ways

3. a. Multiply: $(2x)(5x^2)$
   b. Multiply: $(2x^2)(5x)$
   c. Multiply: $(2)(5x^3)$
   d. Multiply: $(10)(x^3)$
   e. Multiply: $(10x^2)(x)$
   f. Factor $10x^3$ in at least five different ways.
4. Factor $6x^4$ in at least five different ways.
5. Factor $14x^3$ in at least five different ways.
6. What properties of polynomials do you use when you factor monomials?
7. See if you can explain how to factor monomials.
8. What is different about factoring monomials and factoring natural numbers?

To factor a monomial, we find two polynomials (they will be monomials also) whose product is that monomial. Some monomials have many factorizations. One way to factor is to start by factoring the coefficient. Then we can put some of the 'x's with one factor and some of them with the other.

Example:

$$\begin{aligned} 12x^3 \equiv (3 \cdot 4)x^3 &\equiv (3x^2)(4x) \\ &\text{or } (3x)(4x^2) \\ &\text{or } 3(4x^3). \quad \text{[There are still other factorizations]} \end{aligned}$$

EXERCISES

Factor each monomial in at least two ways:

1. $6x^4$
2. $14x^2$
3. $10x^5$
4. $12x^4$
5. $15x^6$
6. $21x^7$
7. 16
8. x^5
9. $3x^4$
10. $6x$

POLYNOMIALS WHOSE TERMS HAVE COMMON FACTORS

Remember how we multiply a polynomial with more than one term by a monomial. We multiply each term by the monomial. This was our definition of multiplication. It is also using the distributive property. By doing the reverse, we can factor.

8.14 Let's Explore

1. a. Multiply: $4(x + 2)$
   b. Factor: $4x + 8$

2. a. Multiply: $x(2x + 3)$
   b. Factor: $2x^2 + 3x$

3. a. Multiply: $4x(x^2 + 3x + 2)$
   b. Factor: $4x^3 + 12x^2 + 8x$

4. a. Multiply: $x^2(2x - 3)$
   b. Factor: $2x^3 - 3x^2$

5. a. Multiply: $3x^2(5x^3 - 4x^2 + 9x - 1)$
   b. Factor: $15x^5 - 12x^4 + 27x^3 - 3x^2$

6. Now let's see how to tell when we can factor this way.
   a. Look at the polynomial $3x^3 + 6x^2 + 9x + 3$. Factor each term separately. Write an equivalent expression, like this:
   $$\_\_\_ \cdot x^3 + \_\_\_ \cdot 2x^2 + \_\_\_ \cdot 3x + \_\_\_ \cdot 1$$
   b. Now use the distributive property to factor.

7. Study this polynomial: $10x^3 + 15x^2 + 35x$
   a. Factor each term separately. Find a factor common to each term.
   b. Factor the polynomial, using the distributive property.
   c. If one of your factors is $2x^3 + 3x^2 + 7x$, there is still a factor x in each term. Factor again. Your answer should be like this:
   $$(\quad)(2x^2 + 3x + 7)$$

8. Study this polynomial: $12x^3 + 30x^2 - 12x$
   Find a factor common to all three terms. Then use the distributive property to factor the polynomial.
   If there is still a factor common to three terms, factor again.

For a polynomial to be factorable by this method there must be a factor common to all its terms. To find the common factor, we factor each term of the polynomial. There is often more than one factor common to all terms. In that case we usually use the one with the largest possible coefficient and the largest exponent. Otherwise we are not factoring completely.

Example 1:

$$\begin{aligned} 12x^3 + 20x^2 &\equiv (4x^2)(3x) + (4x^2)\cdot 5 \\ &\equiv 4x^2(3x + 5) \end{aligned}$$

Example 2:

$$\begin{aligned} 6x^5 - 21x^3 + 3x^2 &\equiv (3x^2)(2x^3) - (3x^2)(7x) + 3x^2 \cdot 1 \\ &\equiv 3x^2(2x^3 - 7x + 1) \end{aligned}$$

If you can see the common factor without writing a factorization of each term, you can write the answer directly.

EXERCISES

Factor completely:

1. $2x + 6$
2. $x^2 - 4x$
3. $3x^2 + 3x$
4. $2x^3 - 3x^2$
5. $x^4 + 2x^3 + 3x^2$
6. $21x^3 - 7x^2 + 7x$
7. $20x^5 + 10x^2 - 15x$
8. $10x^5 - 4x^4 + 6x^3$
9. $10x^6 - 4x^5 + 2x^4 - 6x^3$
10. $35x^5 - 15x^4 + 10x^2 - 5x$
11. $12y^4 - 18y^3 + 15y^2 - 6y$
12. $^{-}15z^6 + 25z^5 - 60z^4$
13. $36x^5 - 60x^4 + 48x^3$
14. $77x^3 + 44x^2 - 55$
15. $7y^5 - 4y^3 + 6y^2 + 5y$
16. $45x^6 - 105x^4 + 60x^2 - 30x$
17. $^{-}39y^5 + 26y^3 - 65y^2$
18. $\frac{3}{2}x^5 - \frac{5}{2}x^3 + \frac{7}{2}x$
19. $\frac{3}{4}y^7 - \frac{7}{4}y^5 + y^3 - 2y^2$
20. $35x^7 - 49x^6 + 14x^5 - 63x^3$

FACTORING TRINOMIALS

When we learned how to multiply binomials quickly, we found that the result was often a trinomial. If we use this method in reverse, we can learn to factor certain trinomials. We cannot factor all trinomials, but we can factor many of them.

8.15 Let's Explore

1. a. Multiply: $(x + 2)(x + 3)$
   b. Factor: $x^2 + 5x + 6$

2. a. Multiply: $(x + 4)(x + 2)$
   b. Factor: $x^2 + 6x + 8$

3. a. Multiply: $(x + 1)(x + 5)$
   b. Factor: $x^2 + 6x + 5$

4. a. Multiply: $(x + 4)(x - 3)$
   b. Factor: $x^2 + x - 12$

5. a. Multiply: $(x + 3)(x - 5)$
   b. Factor: $x^2 - 2x - 15$

6. a. Multiply: $(x - 5)(x - 2)$
   b. Factor: $x^2 - 7x + 10$

7. Now let's see how to tell when we can factor this way.
   a. In each exercise above the coefficient of the x^2 term is what?
   b. Compare the coefficient of the x term and the other term. Use this open sentence to help explain.
   $x^2 + \square\, x + \bigcirc \equiv (x + a)(x + b)$
   c. What must we put in $\square$ to make this true?
   d. What must we put in $\bigcirc$ to make it true?

Now we know how to factor certain trinomials. They must be of this type:

$$x^2 + \square\, x + \bigcirc$$

There must be numbers a and b whose sum is $\square$ and whose product is $\bigcirc$. If we can find such numbers, then we can factor the trinomial as

$$(x + a)(x + b).$$

Example 1:

$$\begin{aligned} x^2 + 5x + 6 &\equiv x^2 + (3 + 2)x + (3 \cdot 2) \\ &\equiv (x + 3)(x + 2) \end{aligned}$$

Example 2:

$$\begin{aligned} x^2 - 3x - 10 &\equiv x^2 + ({}^-5 + 2)x + ({}^-5 \cdot 2) \\ &\equiv (x - 5)(x + 2) \end{aligned}$$

Example 3:

$$\begin{aligned} x^2 - 6x + 8 &\equiv x^2 + ({}^-4 + {}^-2)x + ({}^-4 \cdot {}^-2) \\ &\equiv (x - 2)(x - 4) \end{aligned}$$

You do not need to write all these steps. Learn to write the answer directly.

EXERCISES

Factor:

1. $x^2 + 5x + 4$
2. $x^2 + 7x + 12$
3. $x^2 + 3x - 10$
4. $x^2 + 9x + 18$
5. $x^2 - 3x - 10$
6. $x^2 - 7x + 12$
7. $x^2 - 4x - 5$
8. $x^2 + x - 20$
9. $x^2 - 8x + 12$
10. $x^2 - x - 2$
11. $x^2 - 13x + 12$
12. $x^2 + 12x + 35$
13. $x^2 - x - 12$
14. $x^2 + 18x + 77$
15. $x^2 - 16x + 60$
16. $x^2 + 12x - 45$
17. $x^2 + 10x - 24$
18. $x^2 - 15x + 36$
19. $x^2 + 6x - 7$
20. $x^2 - 8x + 16$
21. $x^2 + 10x + 25$
22. $x^2 - 12x + 36$
23. $x^2 - 15x + 56$
24. $x^2 - x - 56$
25. $x^2 + x - 56$
26. $x^2 + 8x - 9$
27. $x^2 + 14x + 40$
28. $x^2 - 17x + 60$
29. $x^2 - x - 132$
30. $x^2 - 4x - 77$

The trinomials we have factored have all had second degree terms. The coefficient of the second degree term was always 1. What about trinomials where that coefficient is not 1?

8.16 Let's Explore

1. a. Multiply: $(2x + 3)(3x + 1)$
   b. Factor: $6x^2 + 11x + 3$

2. a. Multiply: $(x + 5)(2x + 3)$
   b. Factor: $2x^2 + 13x + 15$

3. a. Multiply: $(3x + 2)(5x + 4)$
   b. Factor: $15x^2 + 22x + 8$

4. a. Multiply: $(2x - 3)(5x + 2)$
   b. Factor: $10x^2 - 11x - 6$

5. a. Multiply: $(5x - 3)(4x - 5)$
   b. Factor: $20x^2 - 37x + 15$

6. Now let's see how to tell when we can factor this way.
   a. In each exercise above we factored a trinomial into two binomials, like this

$$\triangle\, x^2 + \square\, x + \bigcirc \equiv (px + a)(qx + b)$$

   b. What must we put in $\bigcirc$ to make this true?

   c. What must we put in $\triangle$ to make this true?

   d. What must we put in $\square$ to make this true?

Now you can factor certain trinomials. They are of this type:

$$\triangle\, x^2 + \square\, x + \bigcirc$$

There must be numbers p and q whose product is $\triangle$. There must be numbers a and b whose product is $\bigcirc$. Also $a \cdot q + p \cdot b$ must be $\square$. Then we can factor the trinomial as:

$$(px + a)(qx + b)$$

When we factor trinomials like this we must usually try several numbers as coefficients.

Example 1:

If we factor $6x^2 + 7x + 2$, we look for coefficients to fill in the blanks in $(\_\_\_\_x + \_\_\_\_)(\_\_\_\_x + \_\_\_\_)$.
[Let's agree to use only integer coefficients.]

In the first blanks we need numbers whose product is 6. We might try 1 and 6 or 2 and 3.

In the second blanks we need numbers whose product is 2. We might try 1 and 2 or $^{-}1$ and $^{-}2$.

Suppose we try this:
$(3x - 1)(2x - 2)$. When we multiply, we find the middle coefficient is negative. So this choice is not the right one.

Suppose we try
$(3x + 1)(2x + 2)$. When we multiply, we find the middle coefficient is 8 instead of 7. So this choice is not the right one.

Suppose we try
$(3x + 2)(2x + 1)$. When we multiply, we find the middle coefficient is 7, so this is the right choice.

Example 2:

Factor $10x^2 + 11x - 6$.

We look for numbers whose product is 10. There are several possibilities:

1 and 10
$^{-}1$ and $^{-}10$
2 and 5
$^{-}2$ and $^{-}5$

The binomial factors we seek may be like this: $(x + \quad)(10x + \quad)$ or this $(2x + \quad)(5x + \quad)$. We could also try $(^{-}x + \quad)(^{-}10x + \quad)$ or $(^{-}2x + \quad)(^{-}5x + \quad)$, but we don't need to do this. If we can factor with first coefficients negative, we can also do it with first coefficients positive.

Next we look for numbers whose product is $^{-}6$. Again there are several possibilities:

1 and $^{-}6$
$^{-}1$ and 6
2 and $^{-}3$
$^{-}2$ and 3

We try these in different ways until we get two binomials whose product is $10x^2 + 11x - 6$. We know that any of the possibilities listed will give us the correct first and third terms. We need only check the middle term. Let's try $(10x + 1)(x - 6)$. We get the middle term by multiplying the "inside" terms, multiplying the "outside" terms and then adding. When we do, we get $^{-}59x$, so this is not the right combination. Let's try $(5x + 3)(2x - 2)$. This gives a middle term of $^{-}4x$, and we want it to be $11x$, so let's try $(5x - 2)(2x + 3)$. This gives a middle term of $11x$, so we have now factored the polynomial.

In factoring trinomials of this type, you will need some practice in looking for coefficients. Use tables of factors as in example 2 above if you need to. You should learn to find factors without tables, however, as soon as possible. After some practice you will be able to do it rapidly.

EXERCISES

Factor:

1. $2x^2 + 7x + 6$
2. $6x^2 + 9x + 3$
3. $10x^2 + 7x + 1$
4. $2x^2 + 11x + 12$
5. $3x^2 + 7x + 4$
6. $3x^2 + 17x + 10$
7. $3x^2 + 10x + 8$
8. $6x^2 + 11x + 4$
9. $8x^2 + 10x + 3$
10. $10x^2 + 19x + 6$
11. $2x^2 + x - 3$
12. $2x^2 - 7x + 3$
13. $3x^2 + x - 2$
14. $6x^2 - 5x + 1$
15. $2x^2 - 13x + 15$
16. $12x^2 - 17x - 5$
17. $6x^2 - 19x + 10$
18. $2x^2 + 11x - 21$
19. $2x^2 + 13x + 21$
20. $6x^2 + 17x + 7$
21. $15x^2 - x - 2$
22. $12x^2 - x - 6$
23. $12x^2 + 16x - 3$
24. $15x^2 - 14x - 8$
25. $18x^2 + 3x - 10$
26. $18x^2 + 27x + 4$
27. $12x^2 - 7x - 10$
28. $14x^2 - 25x + 6$
29. $20x^2 + 23x + 6$
30. $20x^2 - x - 12$

THE DIFFERENCE OF TWO SQUARES

Now let's look at another of our quick methods of multiplying polynomials. We can learn to use it in reverse.

8.17 Let's Explore

1. a. Multiply: $(x - 3)(x + 3)$
   b. Factor: $x^2 - 9$
2. a. Multiply: $(x + 4)(x + {}^{-}4)$
   b. Factor: $x^2 - 16$
3. a. Multiply: $(2x + 5)(2x - 5)$
   b. Factor: $4x^2 - 25$
4. a. Multiply: $(3x^2 - 5x)(3x^2 + 5x)$
   b. Factor: $9x^4 - 25x^2$

5. a. Multiply: $(5x^3 + 3x^2)(5x^3 - 3x^2)$
   b. Factor: $25x^6 - 9x^4$

6. Now let's see how to tell when we can factor this way.
   a. In exercises 1–5, how many terms are there in each part b?
   b. In each exercise 1–5 part b, list the coefficients of the terms. Explain what pattern you see.
   c. Study the exponents in each exercise 1–5 part b. List them. Explain what pattern you see.
   d. Here is a polynomial like those above. Does it fit the patterns you described?

$$36x^{10} - 25x^4$$

   Factor each term separately. Do it like this:

$$\square \cdot \square - \bigcirc \cdot \bigcirc$$

   e. Now factor into two binomials.

Now you have seen when you can use this method to factor. First, the polynomial must be a binomial. One coefficient must be positive and one negative. Their absolute values must be *squares*, as natural numbers. (This means that we can find a natural number square root.) Also, all exponents must be even. Then we can factor each term into two identical factors.

When a polynomial can be factored into two identical factors, we say it is a *square*, because we can take a square root. The method we are now studying works on any binomial which is the *difference of two squares*:

$$\square^2 - \bigcirc^2$$

We factor such polynomials this way:

$$(\square + \bigcirc)\ (\square - \bigcirc)$$

Example:

$$\begin{aligned} 9x^6 - 49x^4 &\equiv (3x^3)(3x^3) - (7x^2)(7x^2) \\ &\equiv (3x^3 + 7x^2)(3x^3 - 7x^2) \end{aligned}$$

You do not need to write all these steps. Learn to write the answer directly.

EXERCISES

Factor:

1. $x^2 - 25$
2. $4x^2 - 9$
3. $x^6 - 4x^2$
4. $4x^6 - x^2$
5. $49x^4 - 25$
6. $25x^{10} - 4x^6$
7. $36x^4 - 25x^2$
8. $1 - x^2$
9. $x^2 - 36x^6$
10. $25 - 4x^4$
11. $25y^4 - 36y^2$
12. $16 - 25y^6$
13. $4z^2 - 1$
14. $1 - 49z^{12}$
15. $9x^4 - 400$
16. $144x^2 - 1$
17. $81y^8 - 9y^2$
18. $100y^8 - 25$
19. $121x^2 - 144$
20. $225 - 16y^6$

Now you know several ways to factor polynomials. Different types of polynomials are factored differently. You will need to learn to recognize the different kinds. In the next exercises different kinds are mixed up. You will need to look carefully. Decide which kind of polynomial you have, and then factor it.

EXERCISES

Factor:

1. $3x^2 + 12x$
2. $5x^2 + 15x$
3. $4x^2 - 25$
4. $9x^2 - 16$
5. $3x^2 + 15x - 60$
6. $5x^2 + 20x - 15$
7. $x^2 + 2x - 8$
8. $x^2 + x - 6$
9. $3x^3 - 36$
10. $5x^3 - 100$
11. $x^2 + 10x + 25$
12. $x^2 + 12x + 36$
13. $x^2 - 14x + 49$
14. $x^2 - 10x + 25$
15. $4x^2 - 49$
16. $9x^2 - 100$
17. $2x^2 + 5x + 3$
18. $3x^2 + 5x + 2$

19. $21x^2 - 14x$
20. $44x^2 - 33x$
21. $x^2 - 8x + 15$
22. $x^2 + 2x - 15$
23. $x^3 + 3x^2 + x$
24. $x^4 + 4x^2 + 2x$
25. $4x^2 + 4x - 3$
26. $4x^2 - 4x - 3$
27. $25x^2 - 4$
28. $121x^2 - 25$
29. $4x^2 + 4x + 1$
30. $4x^2 + 12x + 9$
31. $4x^2 + 4x + 4$
32. $8x^2 + 8x + 8$
33. $x^2 + 10x + 21$
34. $x^2 + 14x + 45$
35. $25x^2 + 100$
36. $16x^2 + 64$
37. $6x^2 - x - 15$
38. $6x^2 - 7x - 20$
39. $100 - 144x^6$
40. $25 - 121x^8$

SOLVING EQUATIONS BY FACTORING

In the last chapter you already saw how to solve certain equations by factoring. You used the principle of zero products. You did not yet know how to factor polynomials. Now that you know how to factor, you will be able to solve those kinds of equations.

Remember the *principle of zero products*. It says that a product is zero if and only if at least one factor is zero. Suppose we have an equation with '0' for one side, and a polynomial for the other, like this:

$$x^2 + 10x + 16 = 0$$

If we can factor the polynomial on the left, we can use the principle of zero products. In this case we can factor the polynomial:

$$x^2 + 10x + 16 \equiv (x + 2)(x + 8)$$

Since the expressions $x^2 + 10x + 16$ and $(x + 2)(x + 8)$ name the same number for every replacement, we know that the equations

$$x^2 + 10x + 16 = 0 \quad \text{and}$$
$$(x + 2)(x + 8) = 0$$

have the same solution set.

Now we solve the equation.

$$(x + 2)(x + 8) = 0$$
$$x + 2 = 0 \quad \text{or} \quad x + 8 = 0 \text{ (Principle of zero products)}$$
$$x = {}^{-}2 \quad \text{or} \quad x = {}^{-}8$$

The solution set is $\{{}^{-}2, {}^{-}8\}$.

In the following exercises you will use your skill in factoring to solve equations by the principle of zero products.

EXERCISES

Solve these equations. They refer to the system of rational numbers.

1. $x^2 - 9 = 0$
2. $3x^2 + 3x = 0$
3. $x^2 - 4x = 0$
4. $x^2 - 25 = 0$
5. $4x^2 - 9 = 0$
6. $x^2 + 5x + 6 = 0$
7. $x^2 - 2x - 15 = 0$
8. $x^2 - 16x + 60 = 0$
9. $2x^2 - 5x = 0$
10. $x^2 + 12x - 45 = 0$
11. $6x^2 - x - 2 = 0$
12. $8x^2 - 10x - 3 = 0$
13. $x^3 - 8x^2 + 12x = 0$
14. $18x^3 - 32x = 0$

DO YOU KNOW THESE TERMS?

To show that you do know and understand these terms, use each in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

term (319)
similar terms (320)
descending order (322)
ascending order (322)
polynomial (323)
degree of a term (323)
coefficient (323)
"missing" term (324)
monomial (324)
binomial (324)
trinomial (324)
square (358)

CHAPTER SUMMARY

The multiplication dot (317)
"Combining" similar terms (319)
Ascending or descending order (322)
Addition of polynomials (325)
Properties of addition for polynomials (328)
Symbols for additive inverses (330)
Subtraction of polynomials (331)
Multiplication of polynomials (336)
Properties of multiplication for polynomials (341)
Special products of polynomials (342)
Factoring polynomials (349)
Solving equations by factoring (360)

REVIEW EXERCISES

1. Study the phrase:

$$5x^3 + 4x + 2x^2 + {}^{-}3x^2 + 3x^2$$

   a. What are its terms?
   b. What terms are similar?
   c. Combine similar terms and write an expression equivalent to the one above.
   d. Use the distributive property to simplify.

2. Simplify each of the following expressions by "combining similar terms."
   a. $2x^2 + 5x^2$
   b. $3a^2 + 2a + 3a^2 + {}^{-}3a$
   c. $2x^5 + 5x^4 + {}^{-}2x^5$
   d. $7b^3 - 3b^2 + 2b^3 - 5b^2 + 3$
   e. $4x^3 + 2x^5 + x^4 + {}^{-}2x^3 + {}^{-}3x^5 + {}^{-}2$

3. What is a polynomial?

4. What is the *standard* way of writing polynomials?

5. What is the coefficient of the fourth term of the following expression?

$$2x^5 + 3x^4 + {}^{-}5x^3 + {}^{-}6x^2 + 5x + {}^{-}3$$

6. Tell whether each of the following polynomials is a monomial, a binomial, or a trinomial:
   a. $x^2 + 3$
   b. $2x^3 + 3x + 1$
   c. $5x^8$
   d. ${}^{-}x$
   e. $3x^2 - x$

7. Add or subtract polynomials, as shown.
   a. $(x^2 + 2) + (2x^2 + 3)$
   b. $2x + \frac{14}{5}x$
   c. $({}^{-}4x^2 + 2x^3 + 4) + (3x^4 + 2x^2)$
   d. $(3x^2 - 5x + 2) + (2x^2 - 3x - 4)$
   e. $(x + 3) + {}^{-}(x + 3)$
   f. $(2x^2 + 3x + 5) - (3x^2 + 4x + 2)$
   g. $(5x^2 - 3x - 2) - (2x^2 - 5x - 7)$
   h. $(3x^4 + 2x^3 + 3x^2 + 5) + (x^3 - 2x^2 + 3x) - (2x^4 + 5x)$
   i. $(3x^5 + 2x^3) + (5x^2 - 1) - (3x^4 - 2x)$
   j. $(5x^2 + 3) + (10x^2 - 4) - (7x + 3) - (5x^2 - 2x - 4)$

8. Add:

a. $$\begin{array}{rcrcrcr} 5x^3 & + & 3x^2 & + & 2x & + & 5 \\ 7x^3 & & & + & 3x & + & {}^-2 \\ \hline \end{array}$$

b. $$\begin{array}{rcrcrcr} x^5 & + & 3x^4 & & & + & 2 \\ 3x^5 & + & {}^-2x^4 & + & 5x & + & {}^-3 \\ \hline \end{array}$$

9. Subtract:

a. $$\begin{array}{rcrcr} 3a^2 & + & 5a & + & {}^-3 \\ 2a^2 & + & 6a & + & 5 \\ \hline \end{array}$$

b. $$\begin{array}{rcrcrcr} 10a^3 & & & + & 3a & + & 7 \\ 5a^3 & + & 3a^2 & + & {}^-6a & + & {}^-7 \\ \hline \end{array}$$

10. Multiply:

a. $(3a)(5a)$

b. $({}^-x)(5x)$

c. $(x^2)(3x^2)$

d. $({}^-a^3)({}^-a^3)$

e. $(0)(5x^5)$

f. $$\begin{array}{rcrcr} 3x^2 & + & 5x & + & 2 \\ & & 3x & + & 1 \\ \hline \end{array}$$

g. $$\begin{array}{rcr} 2a^2 & + & 5 \\ a & + & {}^-3 \\ \hline \end{array}$$

h. $$\begin{array}{rcrcr} 2b^4 & + & {}^-3b^3 & + & 5 \\ b^2 & + & 2b & + & {}^-3 \\ \hline \end{array}$$

i. $$\begin{array}{rcr} 3x^2 & - & 1 \\ 3x^2 & + & 1 \\ \hline \end{array}$$

j. $$\begin{array}{rcr} 2a & + & 2 \\ 2a & + & 2 \\ \hline \end{array}$$

11. Multiply mentally:

a. $(x + 2)(x + 2)$

b. $(x - 3)(x - 3)$

c. $(5a + 3)(5a + 3)$

d. $(2a - 1)(2a - 1)$

e. $(a - 1)(a + 1)$

f. $(3b + 2)(3b - 2)$

g. $(x + 3)(x + 1)$

h. $(5a - 2)(a - 3)$

i. $(2a - 3)(4a + 5)$

j. $(3x + 5)(2x - 1)$

12. Factor completely:

a. $3x + 9$

b. $x^2 - 5x$

c. $x^2 + 5x + 6$

d. $x^2 - 5x + 6$

e. $x^2 + 7x + 6$

f. $x^2 - 7x + 6$

g. $x^2 + 5x - 6$

h. $x^2 - 5x - 6$

i. $x^2 - 9$

j. $25a^2 - 16$

k. $9a^2 + 6a + 1$

l. $b^2 + 6b + 9$

m. $x^2 - x - 20$

n. $x^2 - 12x - 45$

o. $x^2 - 144$

p. $10x^2 + 19x + 6$

q. $2x^3 + 4x^2 + 2x$

r. $5a^2 - 45$

s. $18x^4 + 3x^3 - 36x^2$

t. $2x^2 - 16x - 18$

13. Solve these equations. They refer to the system of rational numbers.

a. $a^2 - 25 = 0$

b. $2b^2 + 8 = 0$

c. $x^2 - 10x + 9 = 0$

d. $9c^2 + 24c + 16 = 0$

e. $2x^2 - 1250 = 0$

14. Factor into primes and write exponential notation for each of the following:

a. 225
b. 784
c. 96
d. 144
e. 405

15. Simplify each of the following:

a. $\dfrac{\frac{3}{4} - \frac{1}{2}}{^{-}\frac{3}{8} + \frac{1}{8}}$

b. $\dfrac{\frac{2}{3} \cdot (\frac{1}{5} - \frac{2}{10})}{^{-}\frac{5}{3} \cdot (\frac{1}{4} + \frac{2}{3})}$

c. $\dfrac{\frac{5}{16} \cdot (\frac{3}{7} + \frac{25}{14})}{\frac{7}{16} \cdot (\frac{5}{8} - \frac{10}{16})}$

d. $\dfrac{^{-}\frac{1}{5}}{\frac{3}{10}} + \dfrac{^{-}\frac{3}{2}}{^{-}\frac{1}{5}}$

e. $\dfrac{\frac{3}{8} - \frac{1}{2}}{\frac{1}{4}(\frac{2}{3} + {}^{-}\frac{1}{6})} - \dfrac{\frac{1}{3} + \frac{5}{6}}{\frac{2}{3} - \frac{1}{4}}$

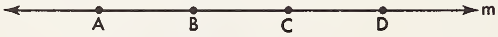
16. $A = \{\cdots {}^{-}4, {}^{-}3, {}^{-}2, {}^{-}1, 0, 1, 2, 3 \cdots\}$
$B = \{\cdots {}^{-}6, {}^{-}4, {}^{-}2, 0, 2, 4, 6, \cdots\}$

a. Find $A \cup B$
b. Find $A \cap B$
c. Find $(A \cup B) \cap B$
d. Find $(A \cap B) \cup A$
e. Find $(A \cup B) \cap (A \cap B)$

17. Draw examples of each of the following:

a. a right isosceles triangle
b. an equilateral triangle
c. a regular hexagon
d. an obtuse scalene triangle
e. a right scalene triangle

18.



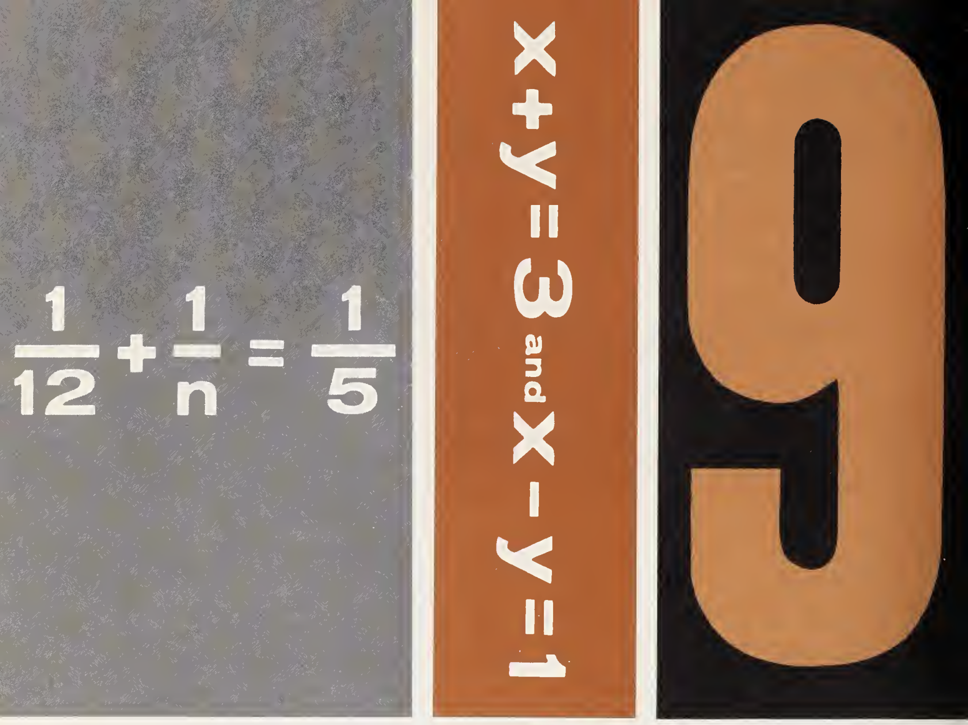
If A, B, C and D are points of a line, as shown:

a. What is the intersection of $\overline{AB}$ and $\overrightarrow{BC}$?
b. What is the intersection of $\overline{AC}$ and $\overline{BD}$?
c. What is the union of $\overset{\circ\!\!\longrightarrow}{BC}$ and $\overset{\circ\!\!\longrightarrow}{BA}$?
d. What is the union of $\overset{\circ\!\!\longrightarrow}{BD}$ and $\overset{\circ\!\!\longrightarrow}{CA}$?
e. What is the intersection of $\overset{\circ\!\!\longrightarrow}{AD}$ and $\overset{\circ\!\!\longrightarrow}{CB}$?

19. Tell what property of the system of rational numbers is illustrated by each of the following:
    a. $^{-}\frac{3}{4} + \frac{3}{5} = \frac{3}{5} + {}^{-}\frac{3}{4}$
    b. $^{-}\frac{5}{6} \cdot {}^{-}\frac{6}{5} = 1$
    c. $^{-}\frac{10}{7} + \frac{10}{7} = 0$
    d. $^{-}\frac{3}{5} \cdot \frac{7}{10} = {}^{-}\frac{21}{50}$
    e. $\frac{6}{7}(\frac{5}{7} + \frac{3}{5}) = (\frac{6}{7} \cdot \frac{5}{7}) + (\frac{6}{7} \cdot \frac{3}{5})$

READING LIST

Lieber, L. *Take A Number.* New York: The Ronald Press Company 1946. pp. 49-54, 104-119.



CHAPTER NINE

Applied Problems, Conjunctions of Equations

$x+y=3$ and $x-y=1 \rightarrow x=2$ and $y=1$

$d=s\cdot t$

6

$x-y=2$ and $x=-y+8 \rightarrow x=5$ and $y=3$

You have been solving applied problems throughout this book. There are many kinds of applied problems, and we use different methods for different kinds. Have you ever wondered whether there is something common to the different kinds of problems? Think about it as you work the problems in the following set. Look for something which is the same about all of them.

EXERCISES

1. A flag has 7 rows of stars, with 11 stars in each row. How many stars are on the flag?
2. You are designing a flag which is to have 45 stars. You want the stars in several rows, with the same number of stars in each row. How many rows can you use?
3. You are designing a flag which is to have 12 stripes. If each stripe is to be 4″ wide, how wide must you make the flag?

4. Mary is wrapping packages for the Christmas party. She knows that it takes about 2 feet of ribbon to tie each package. She must wrap 31 packages. Rolls of ribbon come in 25 foot lengths, 50 foot lengths and 100 foot lengths. How many rolls of each length should she buy?

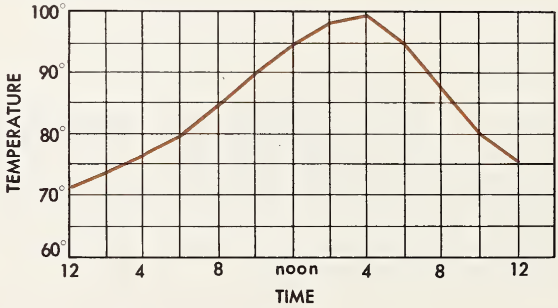
5. Mr. Kane borrows $75 from the bank. The interest rate is 6% per annum. He repays the money 6 months later. How much interest does he pay?

6. Miss Cole's garden is rectangular. It is 20 feet wide and 25 feet long. She wants to put a fence around it. Fencing costs $.65 per foot. How much will it cost to fence the garden?

7. Suppose you live $\frac{1}{2}$ mile from school and it takes you 6 minutes to walk to school. What is your walking speed, in miles per hour?

8. Two boys are on a see-saw. One of them weighs 100 pounds, and is sitting 5 feet from the fulcrum. The other boy weighs 125 pounds. They want to make the see-saw balance. How far from the fulcrum should the second boy sit?

9. This is a graph of the temperature in a factory on July 14. There is a large fan in the factory. It runs whenever the temperature is more than 85°. During what hours on that day was the fan running?



10. Margaret is using a recipe. She wants to make $1\frac{1}{2}$ times the amount of the recipe. Some of the ingredients called for are:

2 cups sugar
4 teaspoons butter
$\frac{3}{4}$ cup flour

How much of each of these ingredients should she use?

Did you find something common to all these problems? It may be that you see more differences than similarities. Answer some questions about these problems, and see if you get a better idea.

9.1 Let's Explore

1. a. For which problems do you need to know how to add, subtract, multiply, divide or factor?
   b. For which problems do you not need to know how to add, subtract, multiply, divide or factor?

2. a. Which problems have just one answer?
   b. Which problems have more than one answer?

3. a. For which problems is a formula helpful?
   b. For which problems is a formula of little or no help?

4. For which problems must you understand the situation before you can solve the problem?

5. For which problems do you use some kind of mathematical language (such as numerals, graphs, etc.) to describe the situation?

6. For which problems do you look for the simplest or most useful way to describe the situation?

7. For which problems do you need to know something besides just mathematics?

You should now begin to see what different applied problems have in common. In each case we must understand the situation. To do this, we must almost always know something besides just mathematics. In each case we describe the situation in mathematical language, using mathematical symbols. Then we look for the simplest or most useful way to re-describe the situation. To do this we use our knowledge of mathematics. We multiply, subtract, factor, simplify, etc. Then from our re-description of the situation we get the answer to the problem.

You should remember that some problems have more than one answer. Some problems have no answer at all. Remember, too, that for some problems we may use formulas, but not all problems can be solved by formula.

To Solve an Applied Problem:

1. *Study and understand the situation.*
2. *Describe the situation in mathematical language.*
3. *Re-describe the situation in more useful mathematical language.*
4. *From the re-description, get the answer to the problem.*

Let's see how these steps are applied in some of the preceding exercises.

In exercise 1 page 367,
The situation: A flag with 7 rows of stars, 11 in each row.
A mathematical description: $7 \cdot 11$ (number of stars)
A re-description: 77.
Answer: 77 stars in the flag.

In exercise 2 page 367,
The situation: A flag with 45 stars, to be arranged in several rows, with the same number in each row.
A mathematical description: 45 (number of stars)
A re-description: $3 \cdot 3 \cdot 5$, or $3 \cdot 15$, or $9 \cdot 5$.
Answers: 3 rows, or 5 rows, or 9 rows, or 15 rows.

In exercise 5 page 368,
The situation: \$75 is borrowed for 6 months at 6%.
A mathematical description:

$$\begin{aligned} I &= P \cdot r \cdot t \text{ and} \\ P &= \$75, \\ r &= 6\% \text{ per annum} \\ t &= \tfrac{1}{2} \text{ yr.} \end{aligned}$$

A re-description: $I = \$75 \cdot 6\% \cdot \frac{1}{2} = \$2.25$.
Answer: \$2.25 interest.

Usually the hardest part of solving applied problems is finding a mathematical description of the situation. You will need to practice it. For a while, let's not solve problems, but only practice making mathematical descriptions.

EXERCISES

In each of the following, do *not* solve the problem. Simply find a mathematical description of the situation.

Example:

In Mr. Norton's garden there are 24 rows of cabbages, and there are 27 cabbages in each row, except for two of the rows. In those two there are only 23 cabbages each. How many cabbages altogether are in the garden?

Number of cabbages:

$$2 \cdot 23 + (24 - 2) \cdot 27$$

1. A square garden is 37 feet on each side. Fence costs $2.63 per foot. There will be a gate 3 feet wide, costing $12.98. How much will it cost to build the fence and put in the gate?

2. A baseball marked $3.98 is on sale at a discount rate of 12%. What is the sale price?

3. Two boys are sitting on a see-saw. One weighs 120 pounds and the other weighs 135 pounds. Where should they sit to balance the see-saw?

4. Merrill is making a swing for his little brother. He has a board for the seat of the swing, but must buy the rope. The swing will be 12 feet long. Rope costs 37¢ per foot. How much will it cost to make the swing?

5. An automobile covers 250 miles of a trip in 5 hours and 25 minutes. At this rate how long will it take for the entire trip of 625 miles?

6. To paint a house 5 men work for three days. Three of them work 8 hours each day. One of them works 8 hours one day and 6 hours each of the other days. The fifth man works 4 hours one day and 7 hours each of the other two days. How many man-hours of work were needed to do the entire job?

7. The population of Mountain Home was 1038 in 1945. It was 2349 in 1950, and increased to 3412 in 1960. What will the estimated population be in 1970?

Now you have practiced describing problem situations in mathematical language. Remember, this is usually the hardest part of solving an applied problem. For this part you usually need to know something besides mathematics. Whenever you work an applied problem, plan to spend the most time on this part of it.

Remember, the second part of solving an applied problem is to re-describe the situation in simpler or more useful mathematical language. To do this you must use what you know about mathematical systems. The third step is to find the answer from your re-description. Now let's practice the second and third steps.

EXERCISES

Use the preceding set of exercises. For each one, re-describe in simpler or more useful language. Then state the answer to the problem.

TRANSLATING TO NUMBER SENTENCES

The first step in solving an applied problem, is to describe the situation in mathematical language. Very often these descriptions are sentences about numbers or measures (called *number sentences*).

You have already worked with number sentences. Equations are number sentences when they refer to number systems or measures. The verb of any equation is $=$. Number sentences may have other verbs, such as $<$ and $\neq$.

You will need practice in describing situations with number sentences.

9.2 Let's Explore

1. Study the sentence "The number is five."
   a. What is the verb of the sentence?
   b. What mathematical verb could we use in place of it to say the same thing?
   c. Write that verb.
   d. Use the variable 'n' to represent 'the number'. Write a number sentence which says the same thing as the English sentence.
2. Write a number sentence for "Her postal zone number is seventeen."
3. Write a number sentence for "The bill is more than 6 dollars."
4. Write a number sentence for "Her change is less than fifty cents."
5. Write a number sentence for "The cost of the movie is not seventy-five cents."
6. What mathematical verb takes the place of the word "is"?
7. What mathematical verb takes the place of "is not"?
8. What mathematical verb takes the place of "is more than"?
9. What mathematical verb takes the place of "is less than"?

In each of the above sentences we could easily write a number sentence using one of the verbs $=$, $>$, $<$ or $\neq$. Did you notice that when you translate to a number sentence, the word "is" by itself becomes $=$? Remember this.

There are other mathematical verbs, such as $\nless$ (is not less than), $\ngtr$ (is not more than), $\leq$ (is less than or equal to), and $\geq$ (is more than or equal to). The verb you will use most often is $=$.

If an English sentence does not say anything about numbers it will have some verb other than one of these, and we have to reword the sentence. We make a new sentence which means the same as the original one, but which says something about numbers. It will then have a verb which can be translated to one of our mathematical verbs. For example, consider the sentence:

"Mary is older than Jane."

We can change the sentence to read,

"Mary's age is more than Jane's age."

This sentence says something about numbers and has the verb *is more than*, so we may translate to a number sentence.

We do have a mathematical symbol for "is more than," so we may translate to a number sentence.

$$A_m > A_j$$

where 'A_m' represents Mary's age and 'A_j' represents Jane's age.

EXERCISES

1. Reword each of the following sentences so that it says something about numbers, and so that you could translate to a number sentence using one of the following verbs:

$$=, \neq, >, \ngtr, <, \nless, \leq, \geq.$$

Example:

John is taller than Bob.
Answer: John's height is more than Bob's height.

a. Mary earned more money than Jane.
b. Sam scored more points than Bill.
c. Bob and Frank are the same size.
d. Sue and Frances are not the same height.
e. The gas bill was not as high this month as it was last month.
f. Paige is older than Bill.
g. Joe earned more than Tom earned.
h. Jane's test grade was lower than Karen's.
i. It is farther from my house to school than from your house to school.
j. It is the same distance from Ed's house to school as it is from his house to Bob's house.

2. Translate each sentence of exercise 1 to a number sentence. We must always think carefully about translating to number sentences. We must be sure to use the proper verb.

9.3 Let's Explore

1. Study the sentence "A certain number is 2 more than 7."
   a. Try to translate to a number sentence, using > for the verb.
   b. Does your sentence say what it should?
   c. Remember that "is" usually translates to '='. Suppose we try to translate this way:

$$\underbrace{\text{A certain number}}_{n}\quad\underbrace{\text{is}}_{=}\quad\underbrace{\text{2 more than 7}}$$

   What mathematical symbols could you use for "2 more than 7"?
   d. Finish translating the sentence.

2. Study the sentence "Twice some number is 3 less than 4."
   a. Try to translate to a number sentence using < for the verb.
   b. Does your sentence say what it should?
   c. Suppose we try to translate this way:

$$\underbrace{\text{Twice some number}}\quad\underbrace{\text{is}}_{=}\quad\underbrace{\text{3 less than 4}}$$

   What would you write for "twice some number"?
   What would you write for "3 less than 4"?
   d. Finish translating the sentence.

3. Study the sentence "Three times a number is 5 more than that number."
   a. Try to translate to a number sentence, using > for the verb.
   b. Does your sentence say what it should?
   c. Suppose we try to translate this way:

$$\underbrace{\text{Three times a number}}\quad\underbrace{\text{is}}_{=}\quad\underbrace{\text{5 more than that number}}$$

   What would you write for "Three times a number"?
   What would you write for "5 more than that number"?
   d. Finish translating the sentence.

4. Study the sentence "One less than a number times one more than the number is 24."
   Let's translate this way:

$$\underbrace{\text{One less than a number}}\quad\underbrace{\text{times}}\quad\underbrace{\text{one more than the number}}\quad\underbrace{\text{is}}\quad\underbrace{\text{24.}}$$

   a. What will you write for "is"?

b. What would you write for "one less than a number"?
Use parentheses. Write a symbol like this:

$$(\_\_\_\_ - 1)$$

c. Write a symbol for "one more than the number."
Use parentheses again, like this:

$$(\_\_\_\_ + 1)$$

d. You should have used the same variable in parts b and c. Why?

e. Finish translating the sentence.

You have now learned more about translating to number sentences. Perhaps you noticed that we most often translate to equations. Now you will need practice in translating.

EXERCISES

Translate each English sentence to a number sentence.

1. A certain number is less than 5.
2. A number is 4 more than $^{-}5$.
3. Twice some number is 3 more than that number.
4. Four times some number is 7 less than that number.
5. A number plus 17 is 2 less than twice that number.
6. One less than a number times 3 more than that number is 12.
7. When a certain number is multiplied by its additive inverse, the result is 25.
8. Five more than a number times that number is 100.
9. Three less than a number times two more than the number is $^{-}5$.
10. One less than twice a number is four more than the number.
11. Twice some number is the additive inverse of that number.
12. If a number is multiplied by itself, the result is twice the number.
13. The product of four less than a number and ten more than the number is twice that number.
14. The product of three less than a number and three more than the number is four more than twice that number.
15. A number plus twice that number is 150.

In these exercises you have translated to number sentences. You have been *describing* situations mathematically. This is the first step in solving an applied problem. You will now be able to use this skill to help solve certain problems.

Example:

> The product of one more than a number and one less than that number is 24. Find the number.

First we describe the situation mathematically, by translating to a number sentence:

$$(n + 1)(n - 1) = 24$$

Now we re-describe. In this case we shall solve the number sentence:

$$(x + 1)(x - 1) = 24$$

Multiply the binomials	$x^2 - 1 = 24$	
Add ${}^{-}24$	$x^2 - 25 = 0$	
Factor	$(x + 5)(x - 5) = 0$	
Use principle of zero products	$x + 5 = 0$	or $x - 5 = 0$
	$x = 5$	or $x = {}^{-}5$

Check:

$$\begin{array}{c|c} (x + 1)(x - 1) & = \quad 24 \\ \hline (5 + 1)(5 - 1) & \\ 6 \cdot 4 & \\ 24 & 24 \end{array} \qquad \begin{array}{c|c} (x + 1)(x - 1) & = \quad 24 \\ \hline ({}^{-}5 + 1)({}^{-}5 - 1) & \\ {}^{-}4 \cdot {}^{-}6 & \\ 24 & 24 \end{array}$$

Both 5 and ${}^{-}5$ check. Both of them satisfy the problem conditions, so we have two answers to the problem.

Finally, we state the answer: 5 or ${}^{-}5$.

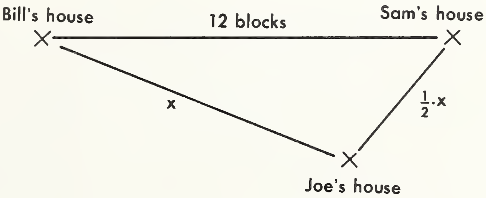
We must be careful to see that we have an answer to the *problem.* Some of the solutions of the equation may not be answers to the problem. Suppose the problem were:

> The product of one more than a certain *number of arithmetic* and one less than that number is 24. Find the number.

We would work the problem the same way. The equation would have the same solutions, 5 and ${}^{-}5$. But ${}^{-}5$ is not an answer to the *problem* because ${}^{-}5$ is not a number of arithmetic. Remember always, any solution of an applied problem must satisfy the conditions of the problem itself.

EXERCISES

Solve the following problems.

1. A certain number plus twice that number is 150. Find the number.
2. A certain number multiplied by four equals 24. Find the number.
3. If you add 4 to a certain number and divide the sum by 6, the result is 2. Find the number.
4. Seventeen added to twice a certain number equals 151. Find the number.
5. Twelve times a number, plus 6 is eighteen times the number, minus six. Find the number.
6. Dan, Bob, and Joe were comparing their earnings for the week. Dan earned twice as much as Joe. Bob earned $2 less than Dan. Altogether they earned $23. How much did each earn?
7. Mary, Jane, Sue and Barbara were selling boxes of candy for their club. Mary sold half as many as Jane. Barbara sold twice as many as Jane. Sue sold three times as many as Mary. The girls sold a total of 40 boxes. How many boxes of candy did each girl sell?
8. 

   Bill walked from his house to Joe's house. He then walked to Sam's house which was half as far as the first distance he walked. If he had walked from his house directly to Sam's he would have walked 3 blocks less. How many blocks is it from Bill's house to Joe's house? from Joe's house to Sam's house?
9. The perimeter of a rectangle is 54″. Its length is 5″ more than its width. What are the dimensions of the rectangle? (Hint: Draw a picture of the situation.)
10. Bob is half as old as his father. Five years from now his father will be 45. How old is Bob?
11. An altitude of a triangle is 14 inches longer than the base, and the area of the triangle is 16 square inches. How long is the base? (Hint: Draw a picture of the situation.)

12. One base of a trapezoid is 8″ shorter than the other. The altitude is the same length as the shorter base. The area of the trapezoid is 5 square inches. How long are the bases?
13. Two boys are balancing on a see-saw. One boy weighs 140 pounds. The other boy weighs 100 pounds and is sitting 2 feet farther from the fulcrum than the first boy. How far from the fulcrum is the first boy sitting? (Hint: Draw a picture of the situation.)

PROBLEMS WITH TOO LITTLE OR TOO MUCH INFORMATION

In the problems you have worked so far you used all the information given. Very often we have information about a situation which we do not need to solve the problem.

Example:

> A train leaves Chicago at 10:00 p.m. on track 9. At 11:00 p.m. it arrives in Joliet. There are 24 cars on the train when it leaves Chicago, and 3 of them are uncoupled in Joliet. If it is 65 miles from Chicago to Joliet, what was the average speed of the train?

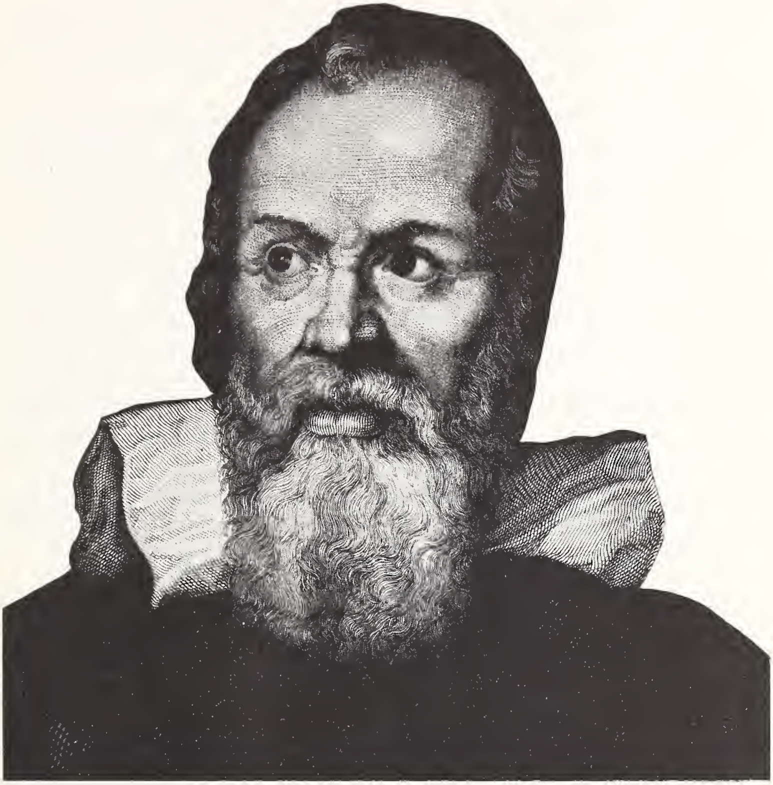
In this example, what is it we wish to find? It is the speed of the train. To find average speed we must know the total distance and the total time for the trip. The problem tells us the distance. It tells us the time the train left and the time it arrived. From this we see that the total time was 1 hr. This is enough information to solve the problem. We do not need to know the track number or the number of cars on the train.

Whenever you work applied problems, you must watch for extra information. If you try to use it you will probably get a silly answer.

EXERCISES

Tell what information is not needed in each of the following problems. Do not solve them.

1. A boat carrying 235 passengers leaves New York, from pier 10 on February 14. It docks at pier 15 in London on February 23. How many days was the boat at sea?
2. Mary earned $6 one week and Jane earned $5. Mary spent half of her money for a book. Jane put $4 of her money in the bank and spent the rest. How much more did Mary earn than Jane?



Galileo Galilei

1564-1642

Around 340 B.C. Aristotle presented in Greece some *reasonable* theories on how objects in the world operate. His laws of physics were found by thinking about the way the world must be. These laws were accepted because it was believed that the same laws which govern our thinking also govern the physical world. No one had bothered to look at the world to see if these laws were true or not. They had to be true.

Nineteen hundred years later Galileo, an Italian, looked at the world and found that some of these Aristotelian laws were not true. He was one of the first men to use the scientific method of looking at the world to see what happens, and then finding a number sentence which describes the situation.

3. Using a number 2 pencil Emma drew a rectangle, 4″ long and 3″ wide. She colored the interior green using a crayon 2 inches long. What is the perimeter of the rectangle?

4. At public school no. 84 the enrollment is 346. This is the second largest school in the third ward. On May 5 only 312 students came to school. How many were absent that day?

5. Mr. Shepard is 40 years old and lives at 383 Madison Avenue. His garden is in the shape of a rectangle 20 feet long and 15 feet wide, and it is 5 feet higher than the street. Fencing costs $1.75 per foot. How much will it cost to fence the garden?

In mathematics class we practice working applied problems. The textbook and the teacher are easily available sources of information. Outside of mathematics class you will have to find your own information. You will need to know how much information is needed to work a problem. To give you practice, this book contains some problems in which there is not enough information given. You will need to figure out what is needed to solve the problem. This makes it more like life outside the classroom.

Sometimes we cannot find all the information needed. What do we do then? About all we can do is make an estimate. Sometimes an estimated answer is good enough.

Example 1:

> Find the perimeter of a polygon if one side is 3″ long and another side is 5″ long.

In this example, we do not know how many sides the polygon has. We know lengths of only two sides. We would need to know the number of sides and the length of each side. In this case we don't have any way to make an estimate.

Example 2:

> The length of a rectangle is 5 feet. What is the area of its interior?

To find the area of the interior of a rectangle, we need to know both length and width. We would need to know the width in this case. We can make a rough estimate this time, however. We know that the width is probably less than the length. If so, it is less than 5 feet. So the area would be less than 25 square feet.

Example 3:

> Hamburgers cost 20¢ each at Joe's Grill. All the students in Mrs. Cooper's class are going to buy a hamburger there for lunch, when they are on a field trip. How much will this cost altogether?

To find the total cost, we must know the number of students and the cost of each hamburger. The missing information is the number of students. In this case we can make an estimate:

There are probably more than 20 students in a class.
There are probably less than 50 students in a class.
Suppose there are 20.
Then the cost is $20 \cdot 20¢ = \$4.00$.
Suppose there are 50.
Then the cost is $50 \cdot 20¢ = \$10.00$.
We have estimated that the cost is between $4.00 and $10.00. The actual answer might be more than $10.00 or less than $4.00, but at least we have some idea about the cost.

EXERCISES

Tell what information is needed to solve each of the following problems. Estimate an answer if it makes sense to do so.

1. Mary earned five dollars more than Jane. How much did Mary earn?
2. Mrs. Williams went to the grocery store to buy potatoes and a loaf of bread. The bread sells for 26¢ per loaf and a five pound bag of potatoes sells for 33¢. How much did Mrs. Williams have to pay for her purchases?
3. Mr. James is paid a salary plus a commission. If his weekly salary is $50, find his total earnings for a week in which he sold $1000 worth of merchandise.
4. Find the perimeter of a rectangle if the length of a side is 4 feet.
5. If cookies sell for 20¢ per pound, how many boxes can be bought for $200?
6. Half as many students were absent today as were absent yesterday. How many students were absent today?
7. Find the perimeter of a triangle if the length of the base is 5 feet and the length of the altitude is 3 feet.
8. Find the total cost of putting a fence around a square lot which is 60 feet on a side.

9. Susan saw a dress in a store window and wanted to buy it. The cost of the dress was $20 and she could not afford to pay that much. Later the store had a discount sale and she was able to purchase the dress. How much did she save by waiting for the sale, to make her purchase?
10. Find the cost of a carpet to cover a rectangular shaped room 20′ × 25′.

CONJUNCTIONS OF SENTENCES

You have already studied one type of sentence important in mathematics. That was the *conditional sentence*. Another type of sentence is also important. Here is an example of one of them:

$$x < 17 \textit{ and } x > 2$$

Notice that two sentences have been joined by the word *and*. Such a compound sentence is called a *conjunction*.

When is a conjunction true? Let's look at the example above. It is neither true nor false until we make a replacement, or *substitution* for 'x'. Which replacements make the sentence true? We see that they are those numbers between 2 and 17. These are the replacements which make *both* of the sentences true.

A conjunction of two sentences is true if and only if both of the sentences are true. Does this fit with the way you use the word "and" ordinarily?

*The **conjunction** of two sentences is another sentence.*
It is formed by joining them with the word "and".
It is true if and only if both sentences are true.

TRANSLATING TO CONJUNCTIONS

Many problem situations can be described in more than one way. Some ways of translating are easier than others. We shall find that many problem situations are quite easy to translate if we use conjunctions of sentences. These sentences will usually be equations.

9.4 Let's Explore

1. Let's translate this problem to mathematical language:
   Bob is half as old as his father. Five years from now his father will be 45. How old is Bob?

a. Think first of the first statement, "Bob is half as old as his father." Use 'x' to represent Bob's age and 'y' to represent his father's age. Translate like this:

$$\underbrace{\text{Bob's age}}_{}\quad \underbrace{\text{is}}_{}\quad \underbrace{\text{half his father's age.}}_{}$$

b. Now let's translate the second statement, "Five years from how his father will be 45." This means that if we add 5 years to his present age, we will get 45. Remember, 'y' represents his present age. Translate like this:

$$\underbrace{\text{His father's age plus 5}}_{}\quad \underbrace{\text{is}}_{}\quad \underbrace{\text{45.}}_{}$$

c. Now take your answers to parts a and b, and write their conjunction, like this:

________________ and ________________.

2. Let's translate this problem to mathematical language:
The sum of two numbers is 65. One number is 21 more than the other. Find the numbers.

a. Translate the first statement, "The sum of two numbers is 65." Use 'a' to represent one of the numbers, and 'b' to represent the other number. Translate like this:

$$\underbrace{\text{The sum of two numbers}}_{}\quad \underbrace{\text{is}}_{}\quad \underbrace{\text{65.}}_{}$$

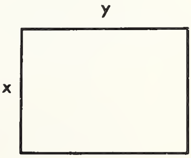
b. Translate the second statement, "One number is 21 more than the other." You used 'a' and 'b' to represent the numbers. You must keep the same symbols. Translate like this:

$$\underbrace{\text{One number}}_{}\quad \underbrace{\text{is}}_{}\quad \underbrace{\text{21 more than the other.}}_{}$$

c. Now write the conjunction of your two equations.

3. Let's translate this problem to mathematical language:
The length of a rectangle is 2 feet more than the width. The perimeter is 28 feet. Find the length and the width.

a. First, draw a picture of the situation. (You should always do this if it helps.)
We may represent the width by 'x' and the length by 'y'. Then the perimeter is $x + x + y + y$, or 2______ + 2______.



b. Now translate the first statement, "The length of the rectangle is 2 feet more than the width."

$$\underbrace{\text{The length of the rectangle}}_{}\quad \underbrace{\text{is}}_{}\quad \underbrace{\text{2 more than the width.}}_{}$$

c. Now translate the second statement, "The perimeter is 28 feet." Remember, for "perimeter" you should not use any new variables.

$$\underbrace{\text{The perimeter}}_{}\quad \underbrace{\text{is}}_{}\quad \underbrace{28.}_{}$$

d. Now write the conjunction of your two equations.

Now you have seen how to translate certain problem situations into conjunctions of equations. First, you should draw a picture when it helps you understand the situation. Next choose two different statements in the problem. Translate each one to a number sentence. Then write the conjunction of those two sentences.

To Translate to a Conjunction:

1. *Draw a picture if it helps you understand the situation.*
2. *Choose two or more statements in the problem.*
3. *Translate each to a number sentence.*
4. *Write the conjunction of those sentences.*

EXERCISES

Translate each of these problem situations to a conjunction of number sentences. Do not try to solve the problems.

1. The sum of two numbers is 62. One of the numbers is 8 more than the other. Find the numbers.
2. If you add 1 to a certain number you get twice another number. The first number is 18 more than the second. Find the numbers.
3. The length of a rectangle is 5 feet more than the width, and its perimeter is 18 feet. Find the length and the width.
4. A certain square has a side 3″ longer than the side of another square. The perimeter of the larger square is 12″ more than the perimeter of the smaller square. How large is each square?
5. A certain rectangle has a perimeter of 28 feet and the area of its interior is 45 square feet. Find its length and its width.
6. John is 3 years older than Frank. In ten years John will be twice as old as Frank. How old are they now?

7. Two boys weighing 90 pounds and 120 pounds are sitting 7 feet apart on a see-saw. How far from the fulcrum is the smaller boy if the see-saw is in balance?

8. Jim has \$3 more than Harry. If Jim had twice as much money he would have \$13 more than Harry. How much money does Harry have?

Now you have had some practice in translating to conjunctions. Remember the next step in solving a problem is to re-describe. To re-describe from a conjunction of equations you must solve them. What do we mean by a *solution* of a conjunction? Let's look at an example:

$$x + y = 8 \quad \text{and} \quad 2x - y = 1$$

Since there are two variables we must find two numbers. One of them will be a replacement for 'x'. The other will be a replacement for 'y'. These replacements must make both sentences true. The numbers 3 and 5 make a solution of this conjunction; 3 for x and 5 for y. Let's check:

$$\begin{array}{c|c} \multicolumn{2}{c}{x + y \quad = \quad 8} \\ \hline 3 + 5 & \\ 8 & 8 \\ & \end{array} \qquad\qquad \begin{array}{r|c} \multicolumn{2}{c}{2x - y \quad = \quad 1} \\ \hline 2 \cdot 3 - 5 & \\ 6 - 5 & \\ 1 & 1 \end{array}$$

The pair of numbers (3, 5) is a solution of the conjunction because it makes both sentences true. We call this an *ordered pair* of numbers, because we have put one of them first and the other second. We agree to use the alphabetical order of the variables. Since 'x' comes before 'y' in the alphabet, we put the number for 'x' before the number for 'y'. The ordered pair (5, 3) is not a solution of the above conjunction. Try it and see.

Remember, when a conjunction has two variables, a solution must consist of an *ordered pair* of numbers, taken in the alphabetical order of the variables.

EXERCISES

See how many of the conjunctions of equations from the last set of problems you can solve. You do not yet know a method, so you will have to guess and try.

SOLVING CONJUNCTIONS OF EQUATIONS

You have probably found that you can solve some conjunctions of equations easily by guessing or by trial. You have probably also found that some of them are not easy to solve. We need to learn a method so that we can solve the harder ones.

9.5 Let's Explore

1. Let's study this conjunction:

$$x + 3 = y \text{ and } y = 5$$

a. What does the sentence $y = 5$ say?

b. If '5' and 'y' are two symbols representing the same number, can we use '5' in place of 'y' wherever we want?

c. Look at the equation $x + 3 = y$. Do you find 'y' in it?

d. Replace 'y' by '5'.

e. You should now have an equation which has 'x' in it, but no other variable. Solve that equation.

f. To solve the conjunction you look for an ordered pair of numbers. What is the first number of the pair (the one that replaces x)?

g. What is the second number of the pair? You can tell at once from the second equation.

h. Then the solution of the conjunction might be the ordered pair (____, ____).

i. Check to see if this is a solution. Substitute for 'x' and 'y' in the conjunction.

2. Let's study the conjunction:

$$x + 2y = 7 \text{ and } x = y + 4$$

a. What does the sentence $x = y + 4$ say?

b. If 'x' and '$y + 4$' represent the same number, can we use '$y + 4$' in place of 'x' wherever we want?

c. Replace 'x' by '$y + 4$' in the first equation.

d. Now you should have an equation with 'y' in it, but no other variable. Solve that equation.

e. You have found the second member of the ordered pair. What is it?

f. Go back to the second equation. Replace 'y' by '1'. You now have an equation with 'x' in it, but no other variable. Solve it.

g. You now have the members of an ordered pair. What is a solution of the conjunction? Be sure to check before you answer.

You have now seen that we can solve some conjunctions of equations by substituting a name from one equation in the other one. We sometimes call this the *method of substitution.* Can you see why? Let's make sure we understand it.

Example:

$$2x + y = 1 \quad \text{and} \quad x = 2y + 8$$

The second equation says that 'x' and '$2y + 8$' represent the same number.

Substitute '$2y + 8$' for 'x' in the first equation.

$$2(2y + 8) + y = 1$$

Now we have an equation with only one variable in it.

Solve it.

$$\begin{aligned} 2 \cdot 2y + 2 \cdot 8 + y &= 1 \\ 5y + 16 &= 1 \\ 5y &= {}^-15 \\ y &= {}^-3 \end{aligned}$$

We need not check here. We will check in both equations later.

Go back to the other equation.

Substitute '${}^-3$' for 'y' and solve:

$$\begin{aligned} x &= 2 \cdot {}^-3 + 8 \\ x &= {}^-6 + 8 \\ x &= 2 \end{aligned}$$

The ordered pair $(2, {}^-3)$ may be a solution. We check:

$$\begin{array}{r|l} \multicolumn{2}{c}{2x + y = 1} \\ \hline 2 \cdot 2 + {}^-3 & \\ 4 + {}^-3 & \\ 1 & 1 \end{array} \qquad \begin{array}{r|r} \multicolumn{2}{c}{x = 2y + 8} \\ \hline & 2 \cdot {}^-3 + 8 \\ & {}^-6 + 8 \\ 2 & 2 \end{array}$$

The solution is $(2, {}^-3)$ because it makes both sentences true.

Now you will need some practice in solving conjunctions. Since you will be writing many of them, you need not write the word "and" every time. Just write this:

$$\begin{aligned} 2x + y &= 1 \\ x &= 2y + 8 \end{aligned}$$

We will understand that this is a conjunction, and that a solution must be an ordered pair that makes both sentences true.

EXERCISES

Solve these conjunctions of equations.

1. $a + b = 8$
   $a = 3b$
2. $x + y = 6$
   $x = y + 2$
3. $2x + y = 7$
   $x = y + 2$
4. $x + 3y = 13$
   $y = 5 - x$
5. $y = 7 + 2x$
   $2x + y = 9$
6. $c = 3d$
   $2c + 2d = 8$
7. $2y + x = 7$
   $y = x - 4$
8. $y = x + 1$
   $3x + y = 17$
9. $2y + 3x = 12$
   $x = 4y - 10$
10. ${}^{-}2x + 4y = {}^{-}18$
    $y = 3x - 17$

In all of the above exercises one of the equations had a variable alone on one side. This made it easy to substitute. In some conjunctions this does not happen. What do we do then?

9.6 Let's Explore

1. Let's study this conjunction:

$$\begin{aligned} x + 2y &= 7 \\ x - y &= 4 \end{aligned}$$

We would like to substitute something for either 'x' or 'y'. Neither equation has 'x' or 'y' alone on one side. If one of them did, we could then see how to substitute.

a. Rewrite the second equation. Now find a new equation by the addition principle. Add y.
b. Now you can substitute for 'x' in the first equation. Do so, and solve the conjunction.

2. Let's study this conjunction:

$$\begin{aligned} 3x + 3y &= 15 \\ 2x - 4y &= {}^{-}8 \end{aligned}$$

Neither equation has 'x' or 'y' alone on one side. If one of them did, we could see how to substitute.

a. Rewrite the second equation.
b. Now find a new equation by the addition principle. Add $4y$.
c. Find another equation by the multiplication principle. Multiply by $\frac{1}{2}$.
d. Now you can substitute for 'x' in the first equation. Do so, and solve the conjunction.

You have now seen that we can use the method of substitution to solve conjunctions, even if neither equation has a variable alone on one side. We choose one of the equations. Then using the addition and multiplication principles, we find an equation having one variable alone on one side. Then we can substitute. Let's make sure we understand this.

Example:

$$\begin{aligned} 3x + 4y &= {}^{-}6 \\ 2x - 4y &= 16 \end{aligned}$$

Let's use the second equation first. From it we will find an equation with 'x' alone on one side:

$$\begin{aligned} 2x - 4y &= 16 \\ 2x - 4y + 4y &= 16 + 4y && \text{(adding } 4y\text{)} \\ 2x &= 16 + 4y && \text{(simplifying left side)} \\ \tfrac{1}{2} \cdot 2x &= \tfrac{1}{2}(16 + 4y) && \text{(multiplying by } \tfrac{1}{2}\text{)} \\ x &= 8 + 2y && \text{(simplifying both sides)} \end{aligned}$$

Sometimes when we do this we say we are "solving the equation for x." (We are not actually solving it, because to solve we find ordered pairs of numbers making it true. This is just a short way of speaking.)

Now we substitute for 'x' in the first equation of the conjunction and then solve the resulting equation.

$$\begin{aligned} 3(8 + 2y) + 4y &= {}^{-}6 \\ 24 + 6y + 4y &= {}^{-}6 && \text{(using distributive property)} \\ 24 + 10y &= {}^{-}6 && \text{(combining similar terms)} \\ {}^{-}24 + 24 + 10y &= {}^{-}6 + {}^{-}24 && \text{(adding } {}^{-}24\text{)} \\ 10y &= {}^{-}30 && \text{(simplifying both sides)} \\ \tfrac{1}{10} \cdot 10y &= \tfrac{1}{10} \cdot {}^{-}30 && \text{(multiplying by } \tfrac{1}{10}\text{)} \\ y &= {}^{-}3 && \text{(simplifying both sides)} \end{aligned}$$

Go back to the other equation. Substitute '${}^{-}3$' for 'y' and solve.

$$\begin{aligned} 2x - 4y &= 16 \\ 2x - 4 \cdot {}^{-}3 &= 16 && \text{(substituting)} \\ 2x + 12 &= 16 && \text{(simplifying)} \\ 2x + 12 + {}^{-}12 &= 16 + {}^{-}12 && \text{(adding } {}^{-}12\text{)} \\ 2x &= 4 && \text{(simplifying)} \end{aligned}$$

The solution of this equation is 2.

The ordered pair $(2, {}^{-}3)$ might be a solution. We check:

$3x + 4y$	$= {}^{-}6$
$3 \cdot 2 + 4 \cdot {}^{-}3$	
$6 + {}^{-}12$	
${}^{-}6$	${}^{-}6$

$2x - 4y$	$= 16$
$2 \cdot 2 - 4 \cdot {}^{-}3$	
$4 - {}^{-}12$	
16	16

The solution is $(2, {}^{-}3)$.

Remember, to solve a conjunction by the method of substitution, "solve" one of the equations for one of the variables. Then substitute for that variable in the other equation. You will now need some practice.

EXERCISES

Solve these conjunctions of equations. They refer to the system of rational numbers.

1. $x + 2y = 7$
$x - y = 4$

2. $x + y = 6$
$x - y = 2$

3. $2a + b = 7$
$a - b = 2$

4. $s + 3t = 13$
$s + t = 5$

5. $2x + y = 9$
$2x - y = 7$

6. $2a - 2b = {}^{-}20$
$2a + 2b = 70$

7. $5e + 6f = {}^{-}32$
$5e - 6f = 52$

8. $2x + 8y = 13$
$2x + 10y = 16$

9. $10a + 30b = {}^{-}20$
$5a + 30b = 26$

10. $5x + 10y = 21$
$10x + 10y = 8$

11. $3x - 7y = {}^{-}36$
$9x - 2y = {}^{-}13$

12. $7x - 5y = {}^{-}1$
$21x + 10y = 17$

13. $16a + 9b = 50$
$4a - 18b = 26$

14. $25x - 15y = {}^{-}40$
$5x - 3y = {}^{-}8$

15. $24s + 8t = 16$
$6s + 2t = 4$

16. $x + y = 7$
$x - y = 7$

17. $2d + e = 14$
$4d + 2e = 20$

18. $a + b = 14$
${}^{-}a + b = 14$

19. $5c + 2d = 10$
$5c + 4d = 10$

20. $2x + 3y = 11$
$3x + 2y = 9$

21. $5x + 7y = 19$
$7x + 5y = 17$

22. $4x - 2y = 2$
$2x - 4y = {}^{-}12$

23. $15x - 5y = 0$
$5x + 15y = 8$

24. $6x + 9y = 8$
$9x - 3y = 1$

25. $8x - 3y = {}^-31$
$2x + 6y = 26$

26. $9x + 7y = 42$
$18x - 2y = {}^-12$

27. $5x - 7y = {}^-45$
$x + 14y = {}^-9$

28. $6x + 10y = 24$
$3x + 5y = 12$

29. $9x - 12y = 6$
$3x - 4y = 6$

30. $5x + 4y = 3$
$5x + 4y = {}^-3$

THE ADDITION METHOD

You have solved conjunctions of equations by the method of substitution. You probably found that sometimes calculations are difficult. There is another method which we shall now learn. It sometimes (but not always) makes calculations easier.

Before we learn this method, let's review the *addition principle* for solving equations which you learned in Chapter 7. Remember what an equation says: The symbols on the two sides of the = sign represent the same number. If we have an equation like $a = b$ and we add any rational number c, we get $a + c = b + c$. If it was true that 'a' and 'b' represent the same number, then it must be true that '$a + c$' and '$b + c$' represent the same thing. There are two reasons we use to get this principle. First, the set of rational numbers is closed under addition. This tells us that '$a + c$' and '$b + c$' represent rational numbers. Second, addition is an *operation*. This means that when we add two numbers we do not get more than one answer. Then if $a = b$ is true, we have just one number. It is named twice, 'a' and 'b'. If we add c to this number there cannot be more than one answer. So '$a + c$' and '$b + c$' must represent the same number. In other words, $a = b \rightarrow a + c = b + c$.

We shall use the *addition principle* as the basis of another method for solving conjunctions.

9.7 Let's Explore

1. Let's study the conjunction:

$$\begin{aligned} x + y &= 3 \\ x - y &= 1 \end{aligned}$$

a. What does the sentence $x + y = 3$ say?

b. If it is true, then what about the following sentence?

$$(x + y) + (x - y) = 3 + (x - y)$$

c. Simplify the left side of this equation.

d. If $x - y = 1$ is true, what is a simpler name for '$x - y$'?

e. Replace '$x - y$' by this simpler name in your answer to c.

f. You should now have an equation with only one variable. Solve it.

g. Substitute in the equation $x + y = 3$ and then solve.

h. Do you now have an ordered pair? Is it a solution of the conjunction? Check to see.

2. Let's study the conjunction:

$$\begin{array}{r} 2x + 3y = \ \ 7 \\ x - 3y = {}^-5 \end{array}$$

a. Suppose $2x + 3y = 7$ is true. Then what about the following sentence?

$$(2x + 3y) + (x - 3y) = 7 + (x - 3y)$$

b. Simplify the left side of this equation.

c. If $x - 3y = {}^-5$ is true, what is a simpler name for '$x - 3y$'?

d. Replace '$x - 3y$' by this simpler name in your answer to b.

e. You should now have an equation with only one variable. Solve it.

f. Substitute in the equation $x - 3y = {}^-5$ and then solve.

g. Do you now have an ordered pair? Is it a solution of the conjunction? Check to see.

3. Let's look again at the conjunction:

$$\begin{array}{r} x + y = 3 \\ x - y = 1 \end{array}$$

We wish to start with the first equation and get another one by adding $x - y$, as before.

a. Write the conjunction like this:

$$\begin{array}{r} x + y = 3 \\ \underline{x - y = 1} \end{array}$$

b. Now add $(x + y)$ and $(x - y)$. Write your answer under the 'x'.

c. Now we want to add 3 and $(x - y)$. But if '1' and '$x - y$' name the same number we can do this by adding 3 and 1. Do this and write your answer under the '1'.

d. Write another = sign under the others. You now have an equation. Solve it.

e. Substitute in the equation $x + y = 3$ and solve. Do you now have a solution of the conjunction?

4. Let's look again at the conjunction:

$$\begin{aligned} 2x + 3y &= 7 \\ x - 3y &= {}^{-}5 \end{aligned}$$

We wish to start with one of the equations and get another one by adding.

a. Write the conjunction like this:

$$\begin{array}{r} 2x + 3y = \phantom{{}^{-}}7 \\ x - 3y = {}^{-}5 \\ \hline \end{array}$$

b. Add $(2x + 3y)$ and $(x - 3y)$.
Write your answer under the 'x'.

c. Now add the same thing to 7 but instead of '$x - 3y$' use '${}^{-}5$'. Write your answer under the '${}^{-}5$'.

d. Write an equals sign between your answers. You now have an equation with one variable. Solve it.

e. Substitute in the equation $x - 3y = {}^{-}5$. Do you now have a solution of the conjunction?

You have now seen examples of the *method of addition* for solving conjunctions of equations. This method is based on the *addition principle* for solving equations. We know, for example, that:

If $x + y = 3$ and $x - y = 1$, *then* $(x + y) + (x - y) = 3 + 1$.

When we add $x + y$ and $x - y$ and simplify we get an expression with only one variable, 'x'. Then we have an equation with only one variable and can solve it. We then substitute in one of the equations of the conjunction. This way we find an ordered pair. If it checks, it is a solution. Let's make sure we understand the method. Study these examples.

Example 1:

$$\begin{array}{rcl} x + {}^{-}3y &=& 7 \\ 2x + 3y &=& 32 \\ \hline 3x + \quad 0 &=& 39 \text{ (adding)} \\ 3x &=& 39 \\ x &=& 13 \end{array}$$

$$\begin{array}{rcl} 13 + {}^{-}3y &=& 7 \\ {}^{-}3y &=& {}^{-}6 \\ y &=& 2 \end{array}$$

Check:

$x + {}^{-}3y$	$= 7$
$13 + {}^{-}3 \cdot 2$	
$13 + {}^{-}6$	
7	7

$2x + 3y$	$= 32$
$2 \cdot 13 + 3 \cdot 2$	
$26 + 6$	
32	32

The solution is (13, 2).

Example 2:

$$\begin{aligned} 5a + 3b &= 11 \\ 2a + {}^{-}3b &= {}^{-}4 \\ \hline 7a &= 7 \quad \text{(adding)} \\ a &= 1 \\ 2 \cdot 1 - 3b &= {}^{-}4 \quad \text{(substituting in the second equation)} \\ 2 + {}^{-}3b &= {}^{-}4 \\ {}^{-}3b &= {}^{-}6 \\ b &= 2 \end{aligned}$$

Check:

$5a + 3b$	$= 11$
$5 \cdot 1 + 3 \cdot 2$	
$5 + 6$	
11	11

$2a - 3b$	$= {}^{-}4$
$2 \cdot 1 - 3 \cdot 2$	
$2 - 6$	
${}^{-}4$	${}^{-}4$

The solution is (1, 2).

EXERCISES

These conjunctions of equations refer to the system of rational numbers. Solve them by the addition method.

1. $\begin{aligned} x + {}^{-}y &= 0 \\ x + y &= 4 \end{aligned}$

2. $\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$

3. $\begin{aligned} {}^{-}2a + y &= {}^{-}9 \\ 2a + 2y &= 12 \end{aligned}$

4. $\begin{aligned} 5c + 3d &= 17 \\ {}^{-}5c + 2d &= 3 \end{aligned}$

5. $\begin{aligned} 10r + 2s &= 10 \\ 6r - 2s &= 6 \end{aligned}$

6. $\begin{aligned} 6a + 2b &= 0 \\ 9a + {}^{-}2b &= 0 \end{aligned}$

7. $$\begin{aligned} 3x - 3y &= 6 \\ 3x + 3y &= 0 \end{aligned}$$

8. $$\begin{aligned} 5s + 2t + 14 &= 0 \\ {}^{-}5s + 2t + {}^{-}6 &= 0 \end{aligned}$$

9. $$\begin{aligned} {}^{-}3b + 2c &= 0 \\ 3b + {}^{-}4c &= {}^{-}1 \end{aligned}$$

10. $$\begin{aligned} {}^{-}\tfrac{3}{2}x + \tfrac{2}{3}y &= 2 \\ \tfrac{3}{2}x + y &= \tfrac{1}{2} \end{aligned}$$

In all of the equations in the previous set of exercises we could easily obtain a new equation with only one variable. This was possible because each equation contained the additive inverse of something in the other equation. This will not be true in every conjunction of equations. For example in this conjunction when we add we get $8x - 2y = 10$.

$$\begin{aligned} 3x + 2y &= 12 \\ 5x + {}^{-}4y &= {}^{-}2 \end{aligned}$$

You might notice that if the '$2y$' were '$4y$' then addition would eliminate the variable 'y'. What can we do in cases like this?

Before we investigate, let's review the *multiplication principle* for solving equations. Remember, it says that from an equation we can get a new one by multiplying. If we have an equation $a = b$ and multiply by c, we get the equation $a \cdot c = b \cdot c$. If the first equation is true, then the second one must also be true. In other words, $a = b \rightarrow a \cdot c = b \cdot c$.

9.8 Let's Explore

1. Let's study the conjunction:

$$\begin{aligned} 3x + 2y &= 12 \\ 5x + {}^{-}4y &= {}^{-}2 \end{aligned}$$

   a. Find a new equation by adding. Does it have just one variable?
   b. Let's try something else. Take the first equation of the conjunction. Find a new one by multiplying by 2.
   c. Write a new conjunction. Use your answer to b for one equation. Use $5x + {}^{-}4y = {}^{-}2$ for the other.
   d. Now find a new equation by adding. Does it have just one variable?
   e. Solve the conjunction of part c.
   f. Is your answer also a solution of the original conjunction?

2. Let's study the conjunction:

$$\begin{aligned} 4x - 12y &= 28 \\ 2x + 3y &= 32 \end{aligned}$$

   a. From the second equation find a new one by multiplying by 4.

- b. Write a new conjunction. Use $4x - 12y = 28$ for one equation. Use your answer to part b for the other.
- c. Now find a new equation by adding. Does it have just one variable?
- d. Solve the conjunction of part c.
- e. Is your answer also a solution of the original conjunction?

3. Let's look again at the conjunction of exercise 2.
    - a. From the second equation, find a new one by multiplying by $^{-}2$.
    - b. Write a new conjunction. Use $4x - 12y = 28$ for one equation. Use your answer to part a for the other.
    - c. Now find a new equation by adding. Does it have just one variable?
    - d. Solve the conjunction of part b.
    - e. Is your answer also a solution of the original conjunction?

You have now seen that we sometimes need the *multiplication principle*. When we have a conjunction and get a new equation by adding, sometimes no variable disappears. When this happens we use the multiplication principle on one of the equations. Then when we add we get an equation with just one variable. Let's make sure we understand. Study these examples.

Example 1:

$$\begin{aligned} 3a + 6b &= {}^{-}6 \\ 5a + {}^{-}2b &= 14 \end{aligned}$$

$$\begin{array}{rcll} 3a + 6b &=& {}^{-}6 & \\ 15a + {}^{-}6b &=& 42 & \text{(multiplying by 3)} \\ \hline 18a &=& 36 & \text{(adding)} \\ a &=& 2 & \\ 3\cdot 2 + 6b &=& {}^{-}6 & \text{(substituting in the first equation)} \\ 6 + 6b &=& {}^{-}6 & \\ 6b &=& {}^{-}12 & \\ b &=& {}^{-}2 & \end{array}$$

Check:

$$\begin{array}{c|c} 3a + \quad 6b & = \quad {}^{-}6 \\ \hline 3\cdot 2 + 6\cdot {}^{-}2 & \\ 6 + {}^{-}12 & \\ {}^{-}6 & {}^{-}6 \end{array} \qquad \begin{array}{c|c} 5a + \quad {}^{-}2b & = \quad 14 \\ \hline 5\cdot 2 + {}^{-}2\cdot {}^{-}2 & \\ 10 + \ 4 & \\ 14 & 14 \end{array}$$

The solution is $(2, {}^{-}2)$.

Example 2:

$$\begin{aligned} 7x + 8y &= 45 \\ 2x + 2y &= 12 \end{aligned}$$

$$\begin{array}{rcrcrl} 7x & + & 8y & = & 45 & \\ {}^-8x & + & {}^-8y & = & {}^-48 & \text{(multiplying by } {}^-4\text{)} \\ \hline {}^-x & & & = & {}^-3 & \text{(adding)} \\ x & & & = & 3 & \text{(multiplying by } {}^-1\text{)} \\ 2 \cdot 3 & + & 2y & = & 12 & \text{(substituting in the first equation)} \\ 6 & + & 2y & = & 12 & \\ & & 2y & = & 6 & \end{array}$$

The solution of this equation is 3.

Check:

$$\begin{array}{rr|r} 7x + & 8y = & 45 \\ \hline 7 \cdot 3 + 8 \cdot 3 & & \\ 21 + & 24 & \\ & 45 & 45 \end{array} \qquad \begin{array}{rr|r} 2x + & 2y = & 12 \\ \hline 2 \cdot 3 + 2 \cdot 3 & & \\ 6 + & 6 & \\ & 12 & 12 \end{array}$$

The solution is (3, 3).

EXERCISES

These conjunctions refer to the system of rational numbers. Solve them.

1. $3s + 3t = 15$
   $2s + 6t = 22$
2. $7b + 2c = 5$
   ${}^-b + 5c = {}^-6$
3. $4a + b = 3$
   $6a + {}^-5b = {}^-2$
4. $3x + 4y = 1$
   $9x + {}^-2y = {}^-4$
5. $5m - 2n = {}^-23$
   ${}^-10m + n = 34$

In all of the conjunctions of the previous exercises we could add and eliminate one variable after using the multiplication principle with one of the equations. This will not always work so well. For example, look at this conjunction:

$$\begin{aligned} 3x + 4y &= 18 \\ 4x + {}^-3y &= {}^-1 \end{aligned}$$

What can we do in cases like this?

9.9 Let's Explore

Study the conjunction:

$$\begin{aligned} 3x + 4y &= 18 \\ 4x + {}^{-}3y &= {}^{-}1 \end{aligned}$$

1. Use the multiplication principle with the first equation.
   - a. Multiply by 2.
   - b. Multiply by 3.
   - c. Multiply by 4.
   - d. Multiply by 5.
   - e. Multiply by 6.
2. Use the multiplication principle with the second equation.
   - a. Multiply by 2.
   - b. Multiply by 3.
   - c. Multiply by 4.
   - d. Multiply by 5.
   - e. Multiply by 6.
3. Look at your answers to exercises 1 and 2. Find a pair of equations (one from each exercise) which have additive inverses in them. Is there more than one such pair?
4. Using the pair of equations from exercise 3, write a conjunction and solve it.
5. Study the conjunction:

$$\begin{aligned} 3x + 2y &= 12 \\ 7x + 5y &= 29 \end{aligned}$$

   - a. Use the first equation. Multiply by 5 to get a new equation.
   - b. Use the second equation. Multiply by ${}^{-}2$ to get a new equation.
   - c. Use your answers to a and b. Write their conjunction.
   - d. Add. Solve that conjunction.
   - e. Is your solution also a solution of the original conjunction?

You have discovered that it is sometimes convenient to find two new equations by multiplying, so that adding will eliminate one variable. Let's make sure we understand. Study these examples.

Example 1:

$$\begin{aligned} 5a + 7b &= {}^{-}9 \\ {}^{-}3a + 5b &= {}^{-}13 \end{aligned}$$

From first equation $15a + 21b = {}^{-}27$ (multiplying by 3)
From second equation ${}^{-}15a + 25b = {}^{-}65$ (multiplying by 5)

$$\begin{array}{rcll}
15a + 21b &=& {}^-27 & \\
{}^-15a + 25b &=& {}^-65 & \\
\hline
46b &=& {}^-92 & \text{(adding)} \\
b &=& {}^-2 & \\
5a + 7 \cdot {}^-2 &=& {}^-9 & \text{(substituting in first equation)} \\
5a + {}^-14 &=& {}^-9 & \\
5a &=& {}^-9 + 14 & \\
5a &=& 5 &
\end{array}$$

The solution of this equation is 1.

Check:

$$\begin{array}{r|r}
5a + 7b & = \ {}^-9 \\
\hline
5 \cdot 1 + 7 \cdot {}^-2 & \\
5 + {}^-14 & \\
{}^-9 & {}^-9
\end{array}
\qquad
\begin{array}{r|r}
{}^-3a + 5b & = \ {}^-13 \\
\hline
{}^-3 \cdot 1 + 5 \cdot {}^-2 & \\
{}^-3 + {}^-10 & \\
{}^-13 & {}^-13
\end{array}$$

The solution is (1, $^-2$).

Example 2:

$$\begin{array}{rcl}
3x + 2y &=& 12 \\
7x + 5y &=& 29
\end{array}$$

$$\begin{array}{lrcll}
\text{From first equation} & 15x + 10y &=& 60 & \text{(multiplying by 5)} \\
\text{From second equation} & {}^-14x + {}^-10y &=& {}^-58 & \text{(multiplying by } {}^-2\text{)} \\
\cline{2-4}
& x &=& 2 & \text{(adding)} \\
& 3 \cdot 2 + 2y &=& 12 & \text{(substituting in first equation)} \\
& 6 + 2y &=& 12 & \\
& 2y &=& 6 &
\end{array}$$

The solution of this equation is 3.

Check:

$$\begin{array}{r|r}
3x + 2y & = \ 12 \\
\hline
3 \cdot 2 + 2 \cdot 3 & \\
6 + 6 & \\
12 & 12
\end{array}
\qquad
\begin{array}{r|r}
7x + 5y & = \ 29 \\
\hline
7 \cdot 2 + 5 \cdot 3 & \\
14 + 15 & \\
29 & 29
\end{array}$$

The solution is (2, 3).

EXERCISES

Use the addition method to find solutions in the system of rational numbers for the following conjunctions of equations.

1. $a + 2b = 3$
   $a + 3b = 4$

2. $3s + 4t = 9$
   $6s + 4t = 14$

3. $6c + 12d = 24$
   $6c + 5d = 17$

4. $5e + 7f = 11$
   $2e + 7f = 17$

5. ${}^{-}3x + 5y = {}^{-}21$
   ${}^{-}3x + 10y = {}^{-}36$

6. $11a + 20b = 9$
   ${}^{-}10a + 20b = 30$

7. $a - 2b = 2$
   $2a - 2b = 0$

8. $3b - 6c = 21$
   $b - 6c = 23$

9. ${}^{-}25x + 15y = {}^{-}40$
   $25x + 15y = 10$

10. ${}^{-}2x + 3y - 19 = 0$
    ${}^{-}2x + 5y - 29 = 0$

11. $17a + 6b = 71$
    $6a - 6b = 90$

12. $7c - 5e + 37 = 0$
    $5c + 5e - 25 = 0$

13. $20b - 7e - 74 = 0$
    $5b + 7e - 1 = 0$

14. ${}^{-}5b = 2a - 31$
    $7b = {}^{-}2a + 53$

15. $c - 11 = 2b$
    $3c - 47 = {}^{-}b$

16. $2a + 3b = 5$
    $4a + 7b = 11$

17. $2x + 3y = 2$
    $8x + 9y = 7$

18. $3a + 2y = 5$
    $15a + 2y = 15$

19. $10c + 10e = 0$
    $20c + 10e = {}^{-}2$

20. $4a + 2b + 4 = 0$
    $8a + 6b + 4 = {}^{-}5$

21. $5c + 15d = {}^{-}10$
    $30d = 2c + 16$

22. $25x - 25y = 0$
    $25x + 25y = 100$

23. $3x + 5y = 30$
    $5x + 3y = 34$

24. $7x + 3y = 23$
    $3y + 7x = 27$

25. $5a + 4b = 9$
    $10a + 12b = 2$

26. $6x = 40y + 34$
    $20y = 9x - 57$

27. $a = 3b - 17$
    $2a = 6 - 2b$

28. $3a + 2b = 18$
    $9a - 54 = {}^{-}6b$

29. $5x - 2y = 6$
    $15x - 20 = 2y$

30. $15x + 2y = 6$
    $25x - 2y = {}^{-}6$

EXERCISES

Use either the substitution method or the addition method to find solutions of the following conjunctions of equations:

1. $x + y = 6$
   $2x + {}^{-}y = 14$

2. $a = 3b$
   $2a + 2b = 16$

3. $3c + 2d = 1$
   $2c + {}^{-}d = 3$

4. $5s + 2t = 24$
   $30s + 12t = 10$

5. $20x + 10y = 40$
   $2x - y = 0$

6. $2a - 3b = 3$
   $4a + 6b = {}^{-}2$

7. $x + 7y - 3 = {}^{-}1$
   $2x - 14y = 0$

8. $a + 3b = 0$
   $20a + {}^{-}15b = 75$

9. $3(x + 2) = y$
   $5x - y = {}^{-}6$

10. $7m + 2n - 12 = 3$
    $6m - n + 3 = 5m$

11. $6a - 2b = 10$
    $9a - 3b = 15$

12. $2x + 7 = y - 3$
    $5x - y = 5$

13. $16b - 5c - 2 = 0$
    $8b + 10c = 6$

14. $3x - y = 7$
    $3x + y = 5$

15. $17a - 34b = 51$
    $2a - 4b = 6$

16. $17a - 34b = 51$
    $2a - 4b = 24$

17. $5m + 2n = {}^{-}10$
    $10m - 3n = {}^{-}160$

18. $a - 3b = 2$
    $6a + 5b = {}^{-}34$

19. $5x + 3y = 2$
    $3x + 5y = {}^{-}2$

20. $7(a - b) = 14$
    $2a = b + 5$

21. $5s + 13t + 11 = 0$
    $13s + 5t - 20 = 9$

22. $6x = 2(5y - 4)$
    $6y = 3(9x - 1)$

23. $5a - 2b + 3 = 1$
    $2a + 5b + 20 = {}^{-}4$

24. $2b - 3c - 9 = 0$
    $4b - 6c - 9 = 0$

25. $3a - 2b = 0$
    $7a + 2b - 6 = {}^{-}6$

26. $2x + y = 0$
    $5x + 2y = {}^{-}5$

27. $7a - 7b = 315$
    $7a + 7b = 77$

28. $6a - 4 = {}^{-}2b$
    $7b + 10a = {}^{-}8$

29. $3x + 2 - 3y = 14$
    $5y - 2x + 7 = {}^{-}7$

30. $20m = 15n - 10$
    $6n = 6 + 9m$

SOME SPECIAL KINDS OF APPLIED PROBLEMS

Let's recall how we use mathematics to solve applied problems. First of all, we need to know something about the world from which the problems come. For almost all applied problems, this means we need to know something besides mathematics.

One example of this is problems about levers, or torques. To work such problems, we must know mathematics. But we must also know something from science—*the law of the lever*. When we work an applied problem we first describe the situation mathematically. In working lever problems, we use the law of the lever to make the mathematical description. Thus lever problems are all somewhat alike, because we use the same scientific law to describe the situations.

Another class of applied problems consists of problems from the business world. Some of these are problems on interest, discount and commission. To describe the situations mathematically we need to know something about the business world—something besides mathematics.

In many applied problems we find the mathematical description easier when we use conjunctions of equations. You have learned how to solve them. Now you can use that knowledge to help you solve certain kinds of applied problems.

As you work these problems, remember the steps in solving an applied problem.

1. Study and understand the situation. (Don't forget to draw a picture, or sketch whenever it helps.)
2. Describe the situation mathematically.
3. Re-describe the situation in more useful mathematical language.
4. From the re-description, get the answer to the problem.

Remember, too, you should watch for unneeded information. Do not try to use any information not necessary. Also watch for problems without enough information. If you find any of these, you might wish to estimate an answer.

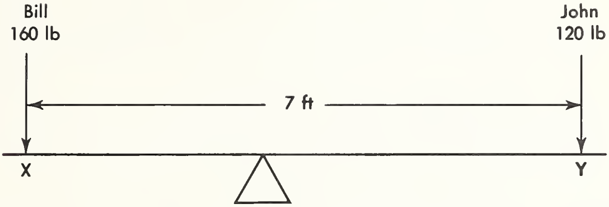
LEVER, OR TORQUE, PROBLEMS

You have worked lever problems before. Let's review the law of the lever. It says that *when forces act on an object, such as a see-saw, a balance occurs if and only if the sum of the torques is zero*. Do you remember the definition of *torque*? We imagine a number line, usually with its origin at the hinge, or *fulcrum*, and perpendicular to the forces. Upward forces are positive and downward forces are negative. To find torques we multiply each force by the number that locates it on the number line.

Example:

> Bill, who is 17 years old and weighs 160 pounds, is sitting 7 feet from John on a see-saw. John is 14 and weighs 120 pounds. The see-saw is balanced. How far is each boy from the fulcrum?

Step 1. Study and understand the situation. Let's draw a picture.



As we do, we see that we do not know where to draw the hinge. So we just draw it somewhere, and call the two distances 'x feet' and 'y feet'.

We can also see that there is unneeded information here. We do not need to know the boys' ages.

If we can find x and y we will have the answer to the problem.

Step 2. Describe the situation mathematically. We know that the law of the lever applies because the see-saw is balanced and not rotating. To describe this situation we will say that the *sum of the torques is zero.*

Let's put the origin at the fulcrum. Then the torques are $^{-}160 \cdot x$ and $^{-}120 \cdot y$. The upward force at the fulcrum produces zero torque. The sum of all torques must be zero.

$$^{-}160x + {}^{-}120y = 0$$

Now we have an equation with two variables. We know we will need another equation to make a conjunction. One equation with two variables usually has too many solutions. Looking at the picture we see that $|x|$ feet $+$ $|y|$ feet $=$ 7 feet, or since we know x is negative and y is positive,

$$^{-}x + y = 7.$$

Step 3. Re-describe the situation.

In this problem we need to know x and y. This means we want to solve the conjunction ${}^{-}160x + {}^{-}120y = 0$ and ${}^{-}x + y = 7$. We may use any method. We will not show the work here, but the solution is the ordered pair $({}^{-}3, 4)$.

Step 4. From the re-description get the answer to the problem.

We look at the picture and see that 'x' gives Bill's location, and 'y' gives John's location. So Bill should be 3 feet to the left of the fulcrum and John 4 feet to the right of it.

Is this an answer to the problem? We must check to see. The torque on Bill's side would be ${}^{-}160$ lb $\cdot$ ${}^{-}3$ ft, or 480 ft-lb. The torque on John's side would be ${}^{-}120$ lb $\cdot$ 4 ft, or ${}^{-}480$ ft-lb. According to the law of the lever the see-saw is balanced. We have an answer to the problem.

In some ways all lever, or torque, problems are alike. For one thing, you use the law of the lever to help translate each of them. On the other hand, no two lever problems are exactly alike. You must use your best imagination and judgment in translating. So we cannot give a rule for every step of every problem. We can say to translate to mathematical language using the law of the lever. We can also give examples. That is about all we can do. Each problem must be translated in the appropriate way, and further rules do not help.

In the following exercises you will see that different problems must be translated differently. In some you will probably translate to a conjunction as in the above example. In others a single equation might be enough. In still others, you may be able to see an answer without an equation.

EXERCISES

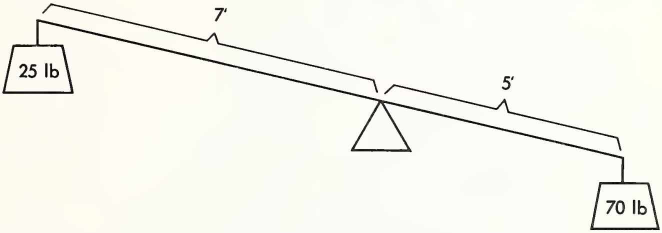
1. How far from a fulcrum must a 2 pound weight be placed to balance a 4 pound weight which is 2 feet from the fulcrum?
2. How far from a fulcrum must a 5 ounce weight be placed to balance a 2 ounce weight which is 10 inches from the fulcrum?
3. A lever is 10 inches long. Where must the fulcrum be placed in order to balance two weights of 3 pounds and 6 pounds respectively, if the weights are at the ends of the lever?
4. How many foot-pounds of torque are needed to move a weight of 1000 pounds, using a lever with its fulcrum 6″ from the 1000 pound weight?

5. a. A man exerting a force of 150 pounds and using a lever 6 feet long would be able to lift a weight of how many pounds if he placed the fulcrum 2 feet from the weight?

   b. The same man could lift how many pounds if he placed the fulcrum 1 foot from the weight?

   c. How many pounds could he lift if the fulcrum was only 6″ from the weight?

6.



In the diagram above where should a 20 pound weight be placed in order to make the lever balance horizontally?

7. The sum of two weights is 45 pounds. A lever is balanced when the larger weight is 3 feet from the fulcrum and the smaller one is 6 feet from the fulcrum. How large are the weights?

8. The difference of two weights is 5 pounds. A lever is balanced when the smaller weight is 6″ from the fulcrum and the larger one is 4″ from the fulcrum. Find the number of pounds in each of the two weights.

9. Two weights, 2′ and 4′ respectively from the fulcrum, balance a weighs 3′ from the fulcrum. The weight 2′ from the fulcrum weighs half at much as the weight 4′ from the fulcrum. The weight 3′ from the fulcrum weighs 10 pounds more than the one 2′ from the fulcrum. How large are the weights?

MOTION PROBLEMS

You have worked motion problems before. These are problems about distances, speeds and times. You should already know the definition of speed:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

For these problems, this definition is the important thing to remember. It is what you will use to help translate.

Let's use letters to represent speed, distance and time.

$$s = \frac{d}{t}$$

Now this definition of speed looks more like we expect a formula to look. Using the multiplication principle for equations we can find another equation equivalent to this one:

$$\begin{aligned} s \cdot t &= \frac{d}{t} \cdot t \\ &= d \cdot \frac{t}{t} \\ &= d \\ \text{so} \qquad d &= s \cdot t \end{aligned}$$

Remember $d = s \cdot t$. It says that distance traveled is speed multiplied by time.

We can also use the multiplication principle to find a formula for time:

$$\begin{aligned} d \cdot \frac{1}{s} &= \frac{1}{s} \cdot s \cdot t \\ \frac{d}{s} &= \frac{s}{s} \cdot t \\ t &= \frac{d}{s} \end{aligned}$$

This formula says that time is distance divided by speed. You need not remember all these. If you remember just one of them (usually we remember $d = s \cdot t$) you can always find the others by the multiplication principle.

Now let's look at some examples of how to work motion problems.

Example 1:

A jet airliner leaves Baltimore, traveling toward San Francisco. At the same time, a propeller-driven plane leaves San Francisco for Baltimore. Both are non-stop flights. The jet flies 600 $\frac{\text{mi}}{\text{hr}}$ and the other airplane flies 300 $\frac{\text{mi}}{\text{hr}}$. How far will the jet fly before it meets the other airplane? The distance from Baltimore to San Francisco is 3000 miles.

Christian Huygens

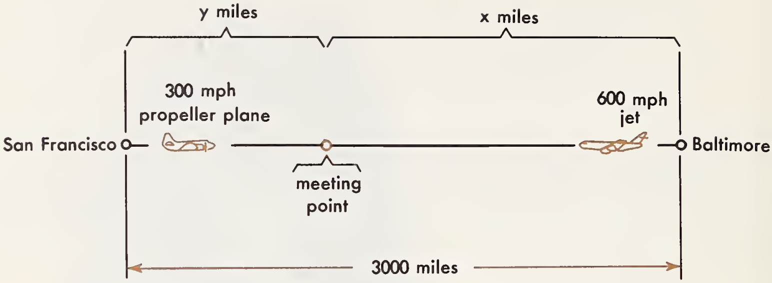
1629-1695

This Dutchman was primarily a physicist. He studied the nature of a swinging pendulum, from which studies came his invention of the pendulum clock.

In his work on the nature of light, he introduced the idea of a wave-front. This idea allowed him to explain in a simple way all the ordinary observations about light. This idea was soon adopted by other physicists who were studying similar observations in heat and sound, etc.

Huygens' book on light looks like a mathematics book. He abstracted the essential form from the physical situation so that it became a problem in geometry. Many properties of geometry were used in this beautiful application.

Step 1. Study and understand the situation. Let's draw a picture.



As we do, we see that we do not know where to draw the point where the two planes meet. So we just draw it in somewhere and call the two distances 'x miles' and 'y miles'. If we can find x we will have an answer to the problem.

Step 2. Describe the situation mathematically.

Since there are distances and speeds involved here we know we are to use the formula $d = s \cdot t$, or else one of the formulas equivalent to it. There are several ways to translate this situation. Here is one way:

Since the airplanes take off at the same time, they fly the same length of time before meeting.

Now $t = \frac{d}{s}$, so

For the fast airplane $t = \frac{x}{600}$

For the slower airplane $t = \frac{y}{300}$

But the times are the same, so

$$\frac{x}{600} = \frac{y}{300}$$

The distances x and y make up the total distance, so

$$x + y = 3000$$

We now have two equations in two variables.

Step 3. Re-describe the situation.

In this problem we need to know x. We may find it by solving the conjunction:

$$\frac{x}{600} = \frac{y}{300} \quad \text{and}$$

$$x + y = 3000$$

We will not show the work here, but the solution is the ordered pair (2000, 1000).

Step 4. From the re-description get the answer to the problem.

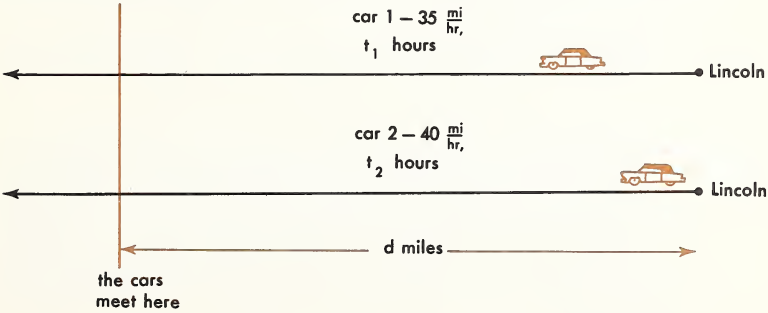
We look at the picture and see that 'x' represents the distance the jet plane flies. This is what we are supposed to find out. So 2000 miles may be an answer to the problem.

Is 2000 miles an answer to the problem? We must check to see. First, if the airplanes fly 2000 miles and 1000 miles, respectively, they will cover the total distance of 3000 miles. This much is all right. Now, the jet flies twice as fast as the other plane. It should go twice as far. From our answer we see that this is the case. The answer checks and we do have an answer to the problem.

Example 2:

A car left Lincoln traveling west at a speed of 35 $\frac{\text{mi}}{\text{hr}}$. An hour later another car left Lincoln traveling in the same direction, at a speed of 40 $\frac{\text{mi}}{\text{hr}}$. The first car has an 80 horsepower engine and the second car has a 120 horsepower engine. How far from Lincoln will the second car overtake the first one?

Step 1. Study and understand the situation. Let's draw a picture.


car 1 — 35 $\frac{\text{mi}}{\text{hr}}$,
t_1 hours
Lincoln
car 2 — 40 $\frac{\text{mi}}{\text{hr}}$,
t_2 hours
Lincoln
d miles
the cars meet here

As we draw the picture, we see that we do not know where to draw the point where the second car catches up. So we just draw it in somewhere, and call the distance 'd miles'. We notice that both cars go the same distance (from Lincoln to the place where the second one catches up).

We know that we will use $d = s \cdot t$. We know the speeds, but do not know the times. We call them 't_1 hours' and 't_2 hours'. If we can find d we will have an answer to the problem.

Step 2. Describe the situation mathematically.

We know that we will use $d = s \cdot t$ for a problem of this kind, or else one of the formulas equivalent to it. Again, there are several possible ways to translate. Here is one way:

Since the cars go the same distance, d is the same for both. For the first car

$$d = 35 \cdot t_1$$

For the second car

$$d = 40 \cdot t_2$$

Then we have

$$35 \cdot t_1 = 40 \cdot t_2$$

This gives us one equation with two variables, 't_1' and 't_2'. We need another. Since car 1 left first it traveled longer. It traveled one hour longer, so t_1 must be 1 more than t_2, or

$$t_1 = 1 + t_2.$$

We now have two equations in two variables.

Step 3. Re-describe the situation.

Let's solve the conjunction. We shall not show the work, but the solution is the ordered pair (8, 7). This means that the first car traveled 8 hours ($t_1 = 8$) and the second traveled 7 hours ($t_2 = 7$).

We still do not know d, but we can find it now, from $d = s \cdot t$.

$$d = 35 \cdot t_1 = 35 \cdot 8 = 280 \text{ miles}$$

Step 4. From the re-description get the answer to the problem.

The answer to the problem is 280 miles, if it checks. The first car goes 280 miles in 8 hours. The second car leaves one hour later, so it travels 7 hours. In 7 hours it goes $40 \cdot 7$ miles, or 280 miles. We see that the cars would then be at the same place. The answer checks and we do have an answer to the problem.

Was there any unneeded information given for this problem?

EXERCISES

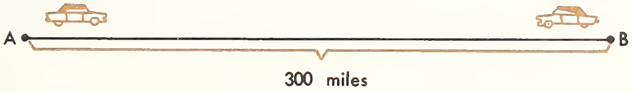
1. A car left Chicago, at a speed of 45 m.p.h. Two hours earlier another car had left Chicago, on the same highway, at a speed of 30 miles per hour. How far from Chicago will the cars be when the second car overtakes the first?

2. A 1961 Rambler and a 1963 Ford leave the same point, at the same time traveling in opposite directions. The Rambler travels at a speed of 60 m.p.h., while the Ford travels at a speed of 48 m.p.h. How far apart will the cars be 3 hours later?

3. Two cars leave Lafayette at the same time traveling in opposite directions on Highway 52. One car travels at a speed of 50 m.p.h. and the other at a speed of 40 m.p.h. The faster car is delayed a half hour because of a flat tire. How far apart are the cars six hours after they departed?

4. Two boys, 10 and 12 years old, leave their respective homes traveling toward one another on their bicycles. The 10 year old boy travels at a speed of 10 m.p.h. while the 12 year old boy travels at a speed of 12 m.p.h. If the two houses are ten miles apart, how far will the 10 year old boy travel before he meets the other boy?

5. A car leaves point A, traveling east at a speed of 45 m.p.h., at the same time another car leaves point B, traveling west at a speed of 35 m.p.h. A and B are 300 miles apart. In how many hours will the two cars meet? Hint: $d = 300$ miles



$s_1 = 45$ m.p.h.
$t =$ no. of hrs. traveling
$d_1 = 45\,t$

$s_2 = 35$ m.p.h.
$t =$ no. of hrs. traveling
$d_2 = 35\,t$

$$d_1 + d_2 = 300$$

6. Two boys agree to ride their bikes to meet each other. One boy travels 8 m.p.h., and the other boy travels 10 m.p.h. The two boys live 10 miles apart. How soon will they meet if they leave at the same time?

7. Mary was going on a 700 mile trip. She missed her bus. The ticket agent told her that the bus leaving 1 hour later made better time and she could catch up with the other one. The first bus averaged only 30 m.p.h. while the second averaged 45 m.p.h. How long would it take the second bus to catch the first one?

8. A 12 horsepower boat travels in still water at a speed of 10 m.p.h. The water in a stream flows at 5 m.p.h.
   a. If the boat is traveling downstream what is its actual speed? (Hint: The boat will travel at its regular speed plus the speed of the water.)
   b. If the boat is traveling upstream, what is its actual speed?
   c. If the boat travels upstream for two hours and then downstream for three hours, where will it then be in relation to its starting point?

9. a. An airplane travels at a speed of 750 miles per hour. If the plane travels against a wind of 50 m.p.h. what is the actual speed of the plane?
   b. How long would it take the plane to fly 1200 miles against a wind of 50 m.p.h.?

10. An airplane with a cruising speed of 780 miles per hour flies with the wind, a distance of 1400 miles. How long does it take the plane to make the trip if the speed of the wind is 20 m.p.h.?

11. A car leaves New York at 8:00 A.M. on Feb. 12, at a speed of 40 m.p.h. A second car leaves at 10:00 A.M. the same day, hoping to catch the first car. At what speed must the second car travel in order to catch the first one at 5:00 P.M. the same day?

12. An airplane leaves Miami traveling toward Washington, a distance of 1500 miles. At the same time another plane leaves Washington traveling toward Miami. The plane flying north travels at a speed of 570 m.p.h. It makes one stop, requiring 40 minutes. The plane flying south travels at a speed of 630 m.p.h. and makes no stops. How long after departure from the respective airports will the two planes meet?

WORK PROBLEMS

We often come across problems about how long it takes to do some work. Problems like these make up another class of applied problems. Like other applied problems, we need to find a principle which enables us to solve them.

9.10 Let's Explore

1. If Joe can complete a job in 2 hours what part of the job can he complete in one hour?

2. a. If he can complete a job in 3 hours, what part of the job can he complete in one hour?
   b. In two hours?

3. a. If he can complete a job in 5 days, what part of the job can he complete in one day?
   b. In two days?
   c. In four days?

4. a. If he can complete a job in d days what part of the job can he complete in one day?
   b. In two days?
   c. In 10 days?

5. a. If one boy can complete a job in 3 days and another can do the same job in 4 days what part can each one do in one day?
   b. What part can they do together in one day?

You have now seen the basic principle by which we can solve work problems. It is this:

> *If a job can be completed in* $\boldsymbol{n}$ *hours (or days), then* $\frac{1}{n}$ *of it can be completed in 1 hour (or day).*

For example, if a man can dig a trench in 6 hours, then he can do $\frac{1}{6}$ of it in one hour. Now let's study an example of how to do such a problem.

Example 1:

Bob can mow a lawn in three hours. It takes Sam four hours to mow the same lawn. If they work together, with two mowers, how long will it take them to mow the lawn?

Step 1. Study and understand the situation.
In this case a picture does not help much, but we can make a list of the facts given and what is to be found.
Bob takes 3 hours alone.
Sam takes 4 hours alone.
Find the time when working together.

Step 2. Describe the situation mathematically.
We use the principle stated above. Since Bob can do the work in 3 hours, he can do $\frac{1}{3}$ of it in one hour. Sam can do $\frac{1}{4}$ of it

in 1 hour. Together they do $\frac{1}{3} + \frac{1}{4}$ of it in 1 hour.
Amount done in 1 hour $= \frac{1}{3} + \frac{1}{4}$.
Also suppose they can mow the lawn together in x hours. Then they can do $\frac{1}{x}$ of the job in 1 hour. So

$$\text{Amount done in 1 hour} = \frac{1}{x}.$$

Now we have

$$\frac{1}{3} + \frac{1}{4} = \frac{1}{x}$$

This is an equation with one variable. If we can find x we will have an answer to the problem, if it checks.

Step 3. Re-describe the situation.
We solve the equation. Since this one is a little different from other equations you have solved, the work is shown here.

$$\begin{aligned}
\frac{1}{3} + \frac{1}{4} &= \frac{1}{x} \\
\frac{1}{3}\cdot\frac{4}{4} + \frac{1}{4}\cdot\frac{3}{3} &= \frac{1}{x} \quad \text{(multiplying by 1)} \\
\frac{7}{12} &= \frac{1}{x} \quad \text{(simplifying)} \\
\frac{7}{12}x &= x\cdot\frac{1}{x} \quad \text{(multiplying by } x\text{)} \\
\frac{7}{12}x &= 1 \quad \text{(property of reciprocals)} \\
x &= \frac{12}{7} \text{ (multiplying by } \tfrac{12}{7}\text{)}
\end{aligned}$$

If we check by substituting '$\frac{12}{7}$' for 'x' in the original equation, we find that it is a solution.

Step 4. From the re-description get the answer to the problem.
To check, we notice that together they do $\frac{7}{12}$ of the job in one hour ($\frac{1}{3} + \frac{1}{4}$). If we multiply by $\frac{12}{7}$, we get 1, which means *all* of the job.

The answer is $1\frac{5}{7}$ hours.

Example 2:

Bob and Bill, working together can do a job in 5 days. Bob could do it alone if he worked for 12 days. How long would it take Bill to do it alone?

Step 1. Study and understand the situation.
Bob and Bill together, 5 days.
Bob alone, 12 days.
Find the time for Bill to do the job alone.

Step 2. Describe the situation mathematically.
Together they do $\frac{1}{5}$ of the job in 1 day.
Bob does $\frac{1}{12}$ of it in 1 day.
Suppose Bill can do it alone in n days.

Then he can do $\frac{1}{n}$ of it in one day.

So in 1 day they do $\frac{1}{12} + \frac{1}{n}$ of the job.

They also do $\frac{1}{5}$ of it. We have an equation.

$$\frac{1}{12} + \frac{1}{n} = \frac{1}{5}$$

Step 3. Re-describe the situation.
We solve the equation:

$$\begin{aligned}
\frac{1}{12} + \frac{1}{n} &= \frac{1}{5} \\
\frac{1}{n} &= \frac{1}{5} - \frac{1}{12} \quad \left(\text{adding } \frac{^{-}1}{12}\right) \\
\frac{1}{n} &= \frac{1}{5} \cdot \frac{12}{12} - \frac{1}{12} \cdot \frac{5}{5} \quad (\text{multiplying by } 1) \\
\frac{1}{n} &= \frac{7}{60} \\
n \cdot \frac{1}{n} &= n \cdot \frac{7}{60} \quad (\text{multiplying by } n) \\
1 &= n \cdot \frac{7}{60} \quad (\text{property of reciprocals}) \\
n &= \frac{60}{7} \quad \left(\text{multiplying by } \frac{7}{60}\right)
\end{aligned}$$

If we check, we find that $\frac{60}{7}$ is a solution of the equation.

Step 4. From the re-description, get the answer to the problem. We check and notice that together they do $\frac{1}{12} + \frac{7}{60}$, or $\frac{1}{5}$ of the job in one day.
If we multiply by 5 we get 1, which means *all* of the job.
The answer is $8\frac{4}{7}$ days.

EXERCISES

1. How many days will it take two boys to do a job if one of them can do the job alone in 15 days and the other needs 12 days to do it alone?
2. If three boys, working separately, can do a job in 5, 6, and 8 days, respectively, how many days will be required for all of the boys to do the job together?
3. One boy can do a job in 6 hours. If another boy helps him, they can do it in 2 hours. How long would the second boy take to do the job alone?
4. A 14 year old boy can do a job in 10 hours. A 15 year old boy can do it in 8 hours. If the first boy works alone for two hours and then the second boy helps him finish, how long will it take them to finish the job?
5. One boy can do a job in 3 hours, a second boy can do the same job in $2\frac{1}{2}$ hours and a third boy requires $3\frac{1}{2}$ hours to do the job. If all three boys work together, how long will it take them to do the job?

DO YOU KNOW THESE TERMS?

To show that you do know and understand these terms, use each one in a sentence so as to illustrate its meaning. If you need help, turn to the page reference listed.

number sentence (372)
equation (372)
conjunction (382)
solution (385)
ordered pair (385)
substitution method (387)
solving for x (389)
addition method (393)
law of the lever (402)
torque (402)
fulcrum (403)
speed (405)

CHAPTER SUMMARY

Translating to number sentences (372)
Problems with too little or too much information (378)
Conjunctions of sentences (382)
Translating to conjunctions (382)
Solving conjunctions of equations (386)
The addition method (391)
Lever, or torque, problems (402)
Motion problems (405)
Work problems (412)

REVIEW EXERCISES

1. Find solution sets for each of the following conjunctions of equations. Use either the substitution method or the addition method. The variables represent rational numbers.

   a. $x + y = 15$
   $x - y = 3$

   b. $2a + b = 15$
   $a + 2b = 27$

   c. $2a + 3c = 2$
   $8a - 3c = 3$

   d. $25c = 15d + 10$
   $20d = 50c - 30$

   e. $^{-}3x = {}^{-}5y - 11$
   $10y = 20 + 20x$

   f. $5a + 10b = 25$
   $a - 5 = {}^{-}2b$

   g. $3x - 10y = 17$
   $6x - 20y = 35$

   h. $15m - 25n + 15 = 0$
   $^{-}6m = 5n$

   i. $20a = 20 - 5b$
   $b = {}^{-}2a$

   j. $7x = 3y - 7$
   $9y = 21(x + 2)$

2. Reword each of the following sentences and then translate each to number sentences.

   a. It is farther from my house to school than it is from your house to school.

   b. Ann and Bob are the same height.

   c. Sue is three years older than Jane.

   d. John's father is five years more than twice as old as John.

   e. Mary lacked by five dollars having enough money to buy the coat.

3. A chartered bus leaves Miami, traveling north at 8:00 A.M., traveling at an average speed of 45 m.p.h. The bus stops for one hour at noon for the passengers to eat lunch. At 10:00 A.M. an automobile leaves Miami, traveling the same route as the bus. If the automobile makes no stops, at what time will it overtake the bus? The average speed of the automobile is 55 m.p.h.

4. How far from the fulcrum must a 120 pound boy sit to balance a boy weighing 100 pounds and sitting 5 feet from the fulcrum?

5. What is the maximum weight a man could move using a six-foot lever with a fulcrum 6 inches from the weight if the man exerts a force of 150 pounds?

6. Mary and June are working together on monthly reports for their employer. Last month Mary did the reports alone and it took her ten hours. The previous month June did them in 8 hours. How long should it take both girls together to do the reports?

7. Mr. Scott has a candy store. One kind of candy sells for 80¢ per pound and another kind of candy sells for 50¢ per pound. Mr. Scott wishes to mix the two kinds together so that the ratio of the cheaper candy to the higher priced one is 2:1. He would like to have 35 pounds of the mixture. How many pounds of each kind of candy should he use?

8. At what price per pound should Mr. Scott sell the candy mixture described in exercise 7 if he wishes to make the same amount of profit when he sells it?

9. Find the products:
   a. $(3a - c)(3a + c)$
   b. $(2a + 7)(2a + 7)$
   c. $(7x + 3y)(8x - 2y)$
   d. $25d^2(6c + 2d)$
   e. $(25 - 3x)(16 - 2x)$

10. Factor completely:
    a. $6a^2 + 13ab + 6b^2$
    b. $81x^2 - 144y^2$
    c. $25c^2 + 100cd + 100d^2$
    d. $x^2 - 2xy + y^2$
    e. $5a^2 - 3ab - 14b^2$

11. Add:
    a. $(2x^3 + 5x + 7 + 10x^4 - 2x^2) + (6x^4 - 3x^5 + 2 - 4x^2)$
    b. $(2ax - 5a^2 + 3b^3) + (7ab - 3ax + 10a^2)$

12. Subtract:
    a. $(7x^3 + 2x - 3) - (5x^2 - 7x + 8)$
    b. $(2a^5 - 3a^3 + 2a^2 + 8) - (7a^5 - 2a^4 + 9a^3 - 7a^2 - 3a + 2)$

13. Solve these equations. They refer to the system of rational numbers.
    a. $2x + 3 = 25$
    b. $20 - 3x = 2$
    c. $ax + b = 0$
    d. $3x + 5 = 2x - 10$
    e. $ax + bx = c$ (Hint: Factor to find the coefficient of x.)
    f. $7x + 21 = 7$
    g. $5x - 2 = 4x$
    h. $(x - 3)(x + 4) = 0$
    i. $3x^2 - 5x = 0$
    j. $x^2 - 7x + 6 = 0$

14. List the properties of the system of natural numbers.

15. What property does the system of whole numbers have that the system of natural numbers does not have?

16. What property does the number system of ordinary arithmetic have that the system of whole numbers does not have?

17. What property does the system of integers have that the system of whole numbers does not have?

18. What properties does the system of rational numbers have that the system of whole numbers does not have?

19. What property does the system of rational numbers have that the system of arithmetic numbers does not have?

20. What property does the system of rational numbers have that the system of integers does not have?

21. Indicate whether each of the following is true or false:
    a. The set of whole numbers is a subset of the set of natural numbers.
    b. The set of arithmetic numbers is a subset of the set of rational numbers.
    c. The set of natural numbers is a subset of the set of integers.
    d. The set of integers is a subset of the set of numbers of ordinary arithmetic.
    e. The set of integers is not closed under the operation of division.
    f. The set of natural numbers is closed under the operations of addition and multiplication.
    g. Zero is the multiplicative identity in the system of whole numbers.
    h. Every rational number has a multiplicative inverse.
    i. Every rational number has an additive inverse.
    j. The set of whole numbers is closed under the operation of subtraction.

22. Graph each of the following:
    a. $x + y = 0$
    b. $x - y = 0$
    c. $y = 2x$
    d. $x = 2y$
    e. $3x - y = 2$

23. Find the square roots of each of the following rational numbers:
    a. 625
    b. 1296
    c. $\frac{256}{81}$
    d. 2.25
    e. .0144

24. Write scientific notation for each of the following:
    a. .00000000892
    b. 345,000,000,000,000
    c. .00000000000000005346
    d. 2,560,000,000,000,000,000,000
    e. .000000000000000001

25. Write simplest exponential notation for each of the following:

a. $5^2 \cdot 5^4$

b. $2^5 \cdot 3^2 \cdot 2^7 \cdot 2^8 \cdot 3^9$

c. $(7^3)^5$

d. $\dfrac{2^3 \cdot 5^7 \cdot 3^5}{3^4 \cdot 2^7 \cdot 5^6}$

e. $\dfrac{1}{3^{-2}}$

f. $(11^{-3})^{-7}$

g. $(13^{-2})^9$

h. $(13^2)^{-9}$

i. $\dfrac{7^2 \cdot 5^3 \cdot 2^9}{3 \cdot 7^2 \cdot 11^5} \cdot \dfrac{11^7 \cdot 3^2}{7^2 \cdot 5} \cdot 19^0$

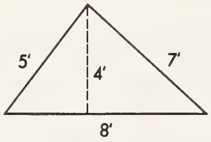
j. $\frac{1}{3} \cdot 3^{-1} \cdot 5^7$

26. Find the total cost of the following purchase:

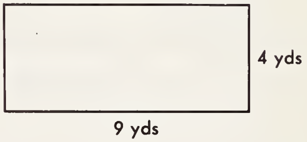
1 jar peach preserves @ 2 for 57¢
2 pounds apples @ 3 pounds for 43¢
3 grapefruit @ 2 for 21¢
1 pound bacon @ 83¢ per pound
2 dozen eggs @ 56¢ per dozen
½ pound candy @ 55¢ per pound

27. Find the perimeter and area of each of the following:

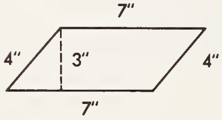
a.



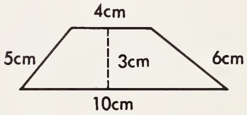
b.



c.

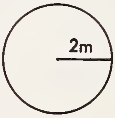


d.

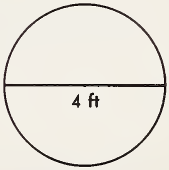


28. Find the circumference and area of each of the following circles: (Use 3.14 for π.)

a.



b.



29. $A = \{0, 1, 2, 3\}$
$B = \{0, 5, 10, 15, 20\}$

a. Draw a Venn diagram to show $A \cup B$

b. Draw a Venn diagram to show $A \cap B$.

30. a. Draw an acute angle.
    b. Use your compass and straightedge to copy the angle you drew.

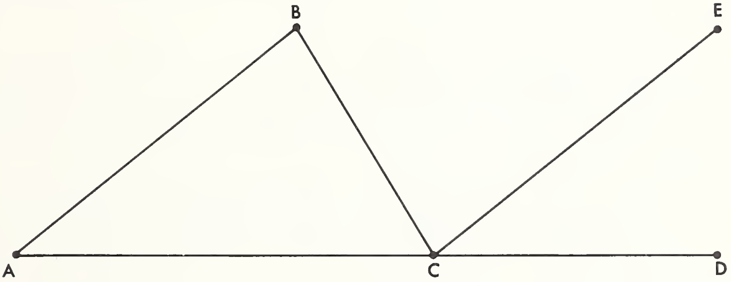
31. a. Draw an obtuse angle.
    b. Use your compass and straightedge to bisect the angle.

32. a. Draw three line segments with lengths 3″, 2″ and 4″ respectively.
    b. Use your compass and straightedge to construct a triangle whose sides are the same length as the segments you drew in a.

33. a. Draw a scalene triangle.
    b. Bisect each side of the triangle.

34. a. Draw an equilateral triangle.
    b. Use your compass and straightedge to construct an altitude of your triangle.

35.



In this figure $\overleftrightarrow{AB} \,||\, \overleftrightarrow{CE}$
Prove: $m\ \angle A + m\ \angle B = m\ \angle BCD$

36. What is another name for a multiplicative inverse?

37. Which of the sentences about integers are true?
    a. $^{+}2 < 5$
    b. $0 > {}^{+}6$
    c. $^{-}5 \leq {}^{+}4$
    d. $|^{-}5| < |^{+}3|$
    e. $|^{-}6| + |^{+}2| \geq |^{+}8|$

38. Write the negation of each of the following sentences.
    a. $^{+}3 + {}^{+}2 = {}^{+}5$
    b. $^{+}3 - {}^{+}6 < {}^{+}2$
    c. $^{-}6 + |^{-}5| > {}^{-}1$
    d. $5 + {}^{-}x > x$
    e. $^{-}3 - {}^{-}4 \neq {}^{+}10$

39. What is another name for a phrase?

40. Tell which of the following pairs of phrases are equivalent.

a. $a - b$ and $b - a$

b. $x - y$ and $x + {}^-y$

c. $^-(^-a)$ and a

d. $\frac{1}{3}$ and 3^{-1}

e. $\frac{a}{b}$ and $\frac{b}{a}$

41. Write the converse of each of the following:

a. If $y < 5$ then $5 > 2$.

b. If $a = {}^-b$ then $|a| = |b|$

42. a. Is exercise 41-b true or false?

b. Is the converse of exercise 41-b true or false?

43. Plot graphs of the following number sentences for integers.

a. $a < b$

b. $x \leq y + 2$

c. $c = |d|$

44. Mr. Lee invested \$500 at 5% interest for two years. The interest is compounded semi-annually. How much is Mr. Lee's investment worth at the end of the two years?

45. Paige bought a pair of ice skates at an end-of-season sale. The marked price of the skates was \$15.95. The dealer was giving a 20% discount on all sales. How much did Paige pay for the skates if there was also a 3% sales tax?

46. Nathan decided to earn some extra money by selling greeting cards. He received a 25% commission. How much money did he make if he sold 6 boxes at \$3.00 each, 10 boxes at \$2.00 each and 12 boxes at \$1.50 each?

47. Find the length of the diagonal of a rectangle if the dimensions of the rectangle are 9 feet by 12 feet.

READING LIST

Lieber, L. *Take A Number*. New York: The Ronald Press Company, 1946. pp. 137-149.

Ruchlis, H. and Engelhardt, J. *The Story of Mathematics*. Irvington-on-Hudson: Harvey House, Publishers, 1958. pp. 117-120, 125-131.

University of Maryland Mathematics Project. *Mathematics for the Junior High School, Second Book*. College Park: Univ. of Md., 1959. pp. 75-87, 92-95, 171-176.

Glossary

This is a list of some important words used in this book. With each a description is given. The description is not necessarily a definition.

A

Absolute value (of a number). The absolute value of a number is the number itself if the number is not negative. The absolute value of a negative number is its additive inverse.

Addition principle. If $a = b$ is a true number sentence, then $a + c = b + c$ is also a true number sentence, for any number c.

Additive inverse. The additive inverse of a number is a number, which when added to the original number, gives zero. If $a + b = 0$, then a and b are additive inverses of each other.

Alternate exterior angles. When two lines are cut by a transversal, several angles are formed. The exterior angles on opposite sides of the transversal are *alternate exterior angles*.

Alternate interior angles. When two lines are cut by a transversal, several angles are formed. The interior angles on opposite sides of the transversal are *alternate interior angles*.

Altitude (of a triangle). A segment from a vertex of a triangle perpendicular to the opposite side.

Angle bisector. A ray which forms two congruent angles with the sides of a given angle. The endpoint of the ray must be the vertex of the angle and the ray must be in the interior of the angle.

Antecedent. The sentence which follows 'if' in a conditional sentence.

Approximate. One number approximates another if the difference between them is small. Each situation determines how close an approximation needs to be.

Area. A number which is the measure of a region.

Ascending order. An arrangement of a polynomial in which the exponent of each term is larger than the one before it.

Associative property. A property of certain operations, stated as follows: $(a * b) * c = a * (b * c)$ for all a, b, c in the system.

Average (of a set of numbers). The sum of the numbers in the set, divided by the number of numbers in the set.

Axis. A number line which is used as a reference in graphing.

B

Base. The number to which an exponent is given. In '5^3', five is the base.

Binomial. A polynomial with two terms.

Bisect. To cut in half.

Boundary. Edge. A line is the boundary of a half-plane. A plane is the boundary of a half-space. A simple closed curve is the boundary of a region.

C

Center of a circle. The point in the interior of the circle which is the same distance from every point of the circle.

Central angle. An angle whose vertex is the center of the circle.

Circle. A set of points which are all the same distance from a given point.

Circumference. A number which is the measure of a circle. The distance around a circle.

Closed segment. A segment which has endpoints.

Closure. A property of certain operations. A set is closed under an operation $*$ if for every two members of the set a and b, $a * b$ is also an element of that set.

Coefficient. A number named in a term of a polynomial. In the polynomial $2x^2 + 9x - 3$, the coefficients are the numbers 2, 9 and 3.

Commutative property. A property of certain operations, stated as follows: $a * b = b * a$ for all a and b in the system.

Compass. An instrument used for transferring distance. It is also used to draw circles.

Concentric circles. Circles that have the same center.

Conditional sentence. An *if* ____ *then* ____ sentence.

Congruent figures. Figures whose corresponding parts have the same measure. Congruent figures have the same size and shape.

Conjunction. A sentence made of two or more sentences joined by the word *and.* A conjunction is true if and only if all parts are true.

Consequent. The sentence following 'then' in a conditional sentence.

Construction. The drawing of geometric figures with straightedge, compass, protractor and other instruments.

Converse (of a conditional sentence). A sentence obtained from the given sentence by interchanging antecedent and consequent. The converse of *If A then B* is the sentence *If B then A.*

Coordinates (of a point). The pair of numbers which locate the point on a graph.

Corresponding angles. When two lines are cut by a transversal several angles are formed. Corresponding angles have the same relative positions about the intersections of the lines.

D

Degree. A unit of measure for angles. A degree is an angle small enough so that 360 of them will fit around a point.

Descending order. An arrangement of a polynomial in which the exponent of each term is smaller than the one before it.

Diameter (of a circle). A segment containing the center of the circle, and whose endpoints are on the circle.

Distributive property. A property of certain pairs of operations, stated as follows: $a*(b\triangledown c) = (a*b)\triangledown(a*c)$, for all a, b and c in the system.

E

Edge. A line which separates a plane into two half-planes is called the *edge* of either half-plane.

Equivalent phrases (or expressions). Phrases which name the same number for all sensible replacements.

Equivalent sentences. Sentences that are true together and false together.

Exponent. A number which tells how many times a number (called the base) is used as a factor. In '5^3' the exponent is three.

Exponential notation. A kind of numeral or number phrase in which exponent symbols are used.

Expression. A number phrase. Any symbol which names a number or *can* name a number for some replacement is an *expression.*

H

Half-plane. A set of all points on a plane which are on one side of a line.

I

Identity. A phrase sentence which states that two phrases are equivalent.

Identity (for an operation). A particular member, a, of a mathematical system for which $a*n =$

n for all n in the system, is an *identity* for the operation $*$.

If and only if. A phrase used to connect two sentences, to form another sentence. The sentence *A if and only if B* means that $A \rightarrow B$ and $B \rightarrow A$.

Inequality. Any number sentence which uses $\neq$, $<$ or $>$ for a verb.

Integers. A set of numbers made up of the whole numbers and their additive inverses.

Inverses (for an operation). If $a*b$ is the identity for the operation $*$, then a and b are inverses of each other.

L

Light-year. A unit of measure for distance. The distance light travels in one year is one light-year.

M

Mega-. A prefix used in the metric system to indicate one million. For example a *megagram* is a million grams.

Micro-. A prefix used in the metric system to indicate one millionth. For example, a *microgram* is a millionth of a gram.

Monomial. A polynomial with just one term.

Multiplication principle. If $a = b$ is a true number sentence then $a \cdot c = b \cdot c$ is also true, for any number c.

N

Negation. A sentence which says that another sentence is false.

Negative number. A number less than zero.

O

Open segment. A segment without endpoints.

Ordered pair. A pair of elements for which one is first and the other second. In the ordered pair of numbers (3, 6) the 3 is first and the 6 is second.

Origin. The point of intersection of the two axes of a graph. It has coordinates (0, 0).

P

Parallel lines. Lines in the same plane which do not meet.

Parallelogram. A quadrilateral with two pairs of parallel sides.

Perpendicular bisector (of a segment). A line containing the midpoint of a segment and perpendicular to the segment.

Polynomial. A number phrase such as $2x^3 + 5x^2 - x + 2$.

Positive number. A number larger than zero.

Power of a number. A number which can be named using exponential notation 'n^p', where p is some whole number. For example, 8 is the third power of 2 because 8 can be named '2^3'.

Proof. An argument for the truth of a statement.

R

Radius. A segment having the center of a circle as one endpoint and a point of the circle as the other endpoint.
Rational numbers. The set of numbers of ordinary arithmetic and their additive inverses.
Reciprocal (of a number). The multiplicative inverse of a number is also called its *reciprocal.*
Rectangle. A parallelogram with four right angles.
Rhombus. A parallelogram having four congruent sides.

S

Scientific notation. A kind of numeral, using exponential notation. Examples are ‘10^4’ and ‘$2.3 \times 10^{^-3}$’.
Semicircle. Half a circle.
Similar terms. Terms of an expression which have the same variable and the same exponent.
Simplify. To find a number phrase equivalent to a given one which, for some reason is accepted as simpler than the given one.
Solution. A number which makes a number sentence true.
Solution set. The set of all solutions of a number sentence is its *solution set.*
Square. A rectangle with four congruent sides.
Square foot. A unit of area. A square region whose sides are 1 foot long.
Speed. The rate at which an object moves. Average speed is found by dividing distance by time.
Straightedge. An instrument for drawing segments.
Substitution. The process of replacing one phrase by another one.
Sweep. The idea of a segment moving across a region. The areas of some regions can be found using sweeps.

T

Term. An addend of a polynomial. In the polynomial $5x^2 + 3x - 7$, the terms are $5x^2$, $3x$ and $^-7$.
Torque. In a lever problem, the product of a force and the number which locates it on the number line.
Transversal. A line which intersects two other lines in the same plane.
Trapezoid. A quadrilateral with at least one pair of parallel sides.
Trinomial. A polynomial with three terms.

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